ASSIGNMENT 1: MEASURE THEORY

(1) Given a nonempty set Ω, describe the smallest and largest σ-algebra of subsets of Ω.

(2) Given $A \subset \Omega$, describe $\mathcal{A}(\{A\})$ and $\mathcal{F}(\{A\})$ the algebra and the σ-algebra generated by the set A.

(3) Let Ω be a nonempty set. Define

$$\mathcal{F}_0 = \{A \subseteq \Omega \mid \text{ either } A \text{ is a finite set or } A^c \text{ is a finite set}\},$$

(assume that $\{\phi\}$ is a finite set). Prove that $\mathcal{F}_0$ is an algebra. But $\mathcal{F}_0$ is not a σ-algebra if Ω is an infinite set.

(4) Let Ω be a nonempty set. Define

$$\mathcal{F}_c = \{A \subseteq \Omega \mid \text{ either } A \text{ is a countable set or } A^c \text{ is a countable set}\}.$$

Prove that $\mathcal{F}$ is a σ-algebra.

(5) Let Ω be a nonempty set and let $\mathcal{C} = \{A_i \mid i \in \mathbb{N}\}$ be a partition of Ω, that is, $A_i \cap A_j = \phi$, for all $i \neq j$ and $\cup_{i \geq 1} A_i = \Omega$. Let

$$\mathcal{F} = \{\cup_{i \in J} A_i \mid J \subset \mathbb{N}\},$$

where for $J = \phi$, $\cup_{i \in J} A_i = \phi$. Prove that $\mathcal{F}$ is a σ-algebra.

(6) Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra on Ω. Foe $A \subset \Omega$ define $\mathcal{F}_A = \{B \cap A \mid B \in \mathcal{F}\}$. Prove that $\mathcal{F}_A$ is a σ-algebra. This is called the *trace σ-algebra of $\mathcal{F}$ on A.*

(7) Consider the semi-algebra $\mathcal{S}$ of semiopen intervals in $\mathbb{R}$ (see Lecture 2). Prove that $\mathcal{F}(\mathcal{S}) = \mathcal{B}$, the Borel σ-algebra.

(8) For $\mathcal{S}$ above, how do we know that $\mathcal{A}(\mathcal{S})$ is not a σ-algebra?

(9) Prove that every countable subset of $\mathbb{R}$ is in $\mathcal{B}$.

(10) Prove that any nonempty open subset of $\mathbb{R}$ is countable union of open intervals (in fact disjoint). The same is true for nonempty open sets of $\mathbb{R}^d$ with intervals replaced by sets of the form $(a_1, b_1) \times \ldots \times (a_d, b_d)$. However, as in $\mathbb{R}$, they may not be disjoint anymore.

(11) For the following collection of subsets $\mathcal{C}_i$, $i = 1, 2, 3$, of $\mathcal{P}(\mathbb{R})$ prove that $\mathcal{F}(\mathcal{C}_i) = \mathcal{B}$.

$$\begin{aligned} \mathcal{C}_1 &= \{(-\infty, x) \mid x \in \mathbb{R}\} \\ \mathcal{C}_2 &= \{(-\infty, x) \mid x \in \mathbb{Q}\} \\ \mathcal{C}_3 &= \{(a, b) \mid a \in \mathbb{Q},\ b \in \mathbb{Q}\} \end{aligned}$$

(12) For the following collection of subsets $\mathcal{C}_i$, $i = 1, \ldots, 4$, of $\mathcal{P}(\mathbb{R}^d)$, $d > 1$, prove that $\mathcal{F}(\mathcal{C}_i) = \mathcal{B}$.

$$\begin{aligned} \mathcal{C}_1 &= \{(a_1, b_1) \times \ldots \times (a_d, b_d) \mid -\infty \le a_i \le b_i \le \infty,\ 1 \le i \le d\} \\ \mathcal{C}_2 &= \{(-\infty, x_1) \times \ldots \times (-\infty, x_d) \mid x_i \in \mathbb{R},\ 1 \le i \le d\} \\ \mathcal{C}_3 &= \{(-\infty, x_1) \times \ldots \times (-\infty, x_d) \mid x_i \in \mathbb{Q},\ 1 \le i \le d\} \\ \mathcal{C}_4 &= \{(a_1, b_1) \times \ldots \times (a_d, b_d) \mid -\infty \le a_i \le b_i \le \infty,\ 1 \le i \le d,\ a_i \in \mathbb{Q}, b_i \in \mathbb{Q}\} \end{aligned}$$

(13) If $\mathcal{C} = \{\{x\} \mid x \in \mathbb{R}\}$ then prove that $\mathcal{F}(\mathcal{C}) = \mathcal{F}_c$ and $\mathcal{F}(\mathcal{C})$ is properly contained in $\mathcal{B}$ (see Problem 4 for the meaning of $\mathcal{F}_c$).

(14) Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ be the extended real line. The Borel σ-algebra on $\bar{\mathbb{R}}$, denoted by $\bar{\mathcal{B}}$, is defined as the smallest σ-algebra containing $\mathcal{B} \cup \{\infty\} \cup \{-\infty\}$. Prove that

$$\bar{\mathcal{B}} = \{A \cup B \mid A \in \mathcal{B},\ B \subseteq \{-\infty, \infty\}\}.$$

(15) Let X and Y be nonempty sets and $f : X \to Y$ be a given function. If $A \subseteq Y$, we define

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\},$$

(even if the inverse function does not exist). If $\mathcal{F}$ is a σ-algebra on Y then prove that

$$\mathcal{F}_f = \{f^{-1}(A) \mid A \in \mathcal{F}\},$$

is a σ-algebra on X.

(16) The aim of this exercise is to prove that any σ-algebra is either finite or uncountable.

(a) Suppose X is a nonempty set and $\mathcal{F}$ is a σ-algebra on X. If $\mathcal{F}$ is an infinite set prove that so is X.

(b) Suppose X is countably infinite set and so is $\mathcal{F}$. Define $f : X \to \mathcal{F}$ by

$$f(x) = \cap_{x \in A \in \mathcal{F}} A, \qquad x \in X,$$

Prove that $f(x)$ is the smallest set in $\mathcal{F}$ containing x.

(c) If $x \in X$, $y \in X$ and $f(x) \cap f(y) \neq \phi$ then prove that $f(x) = f(y)$. Conclude that $f(X)$ (which is a subset of $\mathcal{F}$) is a partition of X.

(d) If $A \in \mathcal{F}$ then prove that

$$A = \cup_{x \in A} f(x).$$

(e) If X is an infinite set then prove that $f(X)$ is an infinite subset of $\mathcal{F}$.

(f) If X is an infinite set and $\mathcal{F}$ is an infinite set then $\mathcal{F}$ must be uncountable.