ASSIGNMENT 2: MEASURE THEORY

1. Consider $(\Omega, \mathcal{A}, \nu)$, where $\mathcal{A}$ is an algebra over Ω and ν is a premeasure. Recall that ν^* is the outer measure on $\mathcal{P}(\Omega)$ induced by ν and $\mathcal{M}$ is the corresponding sigma algebra, that is,

$$\mathcal{M} = \{E \subset \Omega : \nu(A) = \nu(A \cap E) + \nu(A \cap E^c),\ \forall A \subset \Omega\}.$$

   (a) If $\nu^*(E) = 0$, then prove that $E \in \mathcal{M}$.

   (b) Prove that if $\nu^*(A) = 0$, then $\nu^*(A \cup B) = \nu^*(B)$ for any $B \subseteq \Omega$.

   (c) If $F \in \mathcal{M}$ and $\nu^*(F \triangle E) = 0$, then prove that $E \in \mathcal{M}$ (here $F \triangle E = (F \setminus E) \cup (E \setminus F)$).

   (d) Prove that for any $A \subseteq \Omega$ there exists $B \in \mathcal{F}(A)$ such that $A \subseteq B$ and $\nu^*(A) = \nu^*(B)$.

2. Consider a measure space $(\Omega, \mathcal{F}, \mu)$.
   If $\{E_i\}_{i=1}^{\infty} \subset \mathcal{F}$, define

$$\limsup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} E_i, \qquad \liminf E_i = \bigcup_{n=1}^{\infty} \bigcap_{i \geq n} E_i.$$

   (a) Prove that

$$\limsup E_i = \{\omega \in \Omega \mid \omega \in E_i \text{ for infinitely many } i\},$$
$$\liminf E_i = \{\omega \in \Omega \mid \omega \in E_i \text{ for all but finitely many } i\}.$$

   (b) What is the relationship between $\chi_{\limsup E_i}$ and $\limsup_{i\to\infty} \chi_{E_i}$? Ask a similar question about $\chi_{\liminf E_i}$ and $\liminf_{i\to\infty} \chi_{E_i}$.

   (c) Let $\{E_i\} \subset \mathcal{F}$ with

$$\sum_{i \geq 1} \mu(E_i) < \infty.$$

   Prove that

$$\mu\big(\{\omega \in \Omega \mid \omega \in E_i \text{ for infinitely many } i\}\big) = 0.$$

   This result is known as the Borel–Cantelli lemma.

3. If $E \subset \mathbb{R}$ is countable, then prove that $\lambda(E) = 0$, where λ denotes the Lebesgue measure.

4. Let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of $\mathbb{R}$ and define

$$G = \bigcup_{n=1}^{\infty} \left(d_n - \frac{1}{n^2},\ d_n + \frac{1}{n^2}\right).$$

   Prove that for every closed set $F \subset \mathbb{R}$,

$$\lambda(G \triangle F) > 0.$$

5. Consider the nondecreasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Define λ_F in the usual way. Find λ_F^* and $\mathcal{M}_F$ explicitly.

6. Consider $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$ with λ being the Lebesgue measure on $\mathbb{R}$. If $A \in \mathcal{B}_{\mathbb{R}}$ and $x_0 \in \mathbb{R} \setminus \{0\}$, then prove that
   (a) $A + x_0 = \{a + x_0 \mid a \in A\} \in \mathcal{B}_{\mathbb{R}}$.
   (b) $-A = \{-a \mid a \in A\} \in \mathcal{B}_{\mathbb{R}}$.
   (c) $x_0 A = \{x_0 a \mid a \in A\} \in \mathcal{B}_{\mathbb{R}}$.
   (d) $\lambda(A + x_0) = \lambda(A)$ and $\lambda(x_0 A) = |x_0|\lambda(A)$.

7. Suppose μ is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu(A + x) = \mu(A)$ for all $A \in \mathcal{B}$ and all $x \in \mathbb{R}$. If $\mu([0, 1]) = 2$, then prove that $\mu = 2\lambda$.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $\{A_\alpha\}_{\alpha\in\Lambda} \subset \mathcal{F}$ be a disjoint collection of sets of positive measure. Prove that Λ is a countable set. Show by an example that the result is false without σ-finiteness.

9. Let $E \subset \mathbb{R}$ with $\lambda^*(E) = 0$, where λ^* is the Lebesgue outer measure on $\mathbb{R}$. Prove that E^c is dense in $\mathbb{R}$. Is the same true if λ is replaced by λ_F?

10. Suppose $E \in \mathcal{L}$ with $0 < \lambda(E) < \infty$. Given any $\alpha \in (0, 1)$ prove that there exists $E_\alpha \in \mathcal{L}$ such that $E_\alpha \subset E$ and $\lambda(E_\alpha) = \alpha\lambda(E)$. Is the same result true for any λ_F? Here, $\mathcal{L}$ is the σ-algebra of Lebesgue measeurable sets in $\mathbb{R}$. (Hint: Think about the function $f_E(x) = \lambda(E \cap (-\infty, x])$, $x \in \mathbb{R}$.)

11. Suppose $E \in \mathcal{L}$ with $\lambda(E) = \infty$. Given any $\alpha \in [0, \infty)$ prove that there exists $E_\alpha \in \mathcal{L}$ such that $E_\alpha \subset E$ and $\lambda(E_\alpha) = \alpha$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, that is, $(\mathcal{L}, \mathcal{L})$-measurable, where $\mathcal{L}$ is Lebesgue σ-algebra over $\mathbb{R}$. Prove that there exists a set of positive Lebesgue measure on which f is bounded.

13. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f|$ is Lebesgue measurable but f is not.

14. If $f : \mathbb{R} \to \mathbb{R}$ is monotone then f is Borel measurable, that is, $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable.

15. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, then prove that f' is Borel measurable.

16. Let $(\Omega, \mathcal{F})$ be a measurable space and $f : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then prove that $g \circ f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable.

17. If $f : \Omega \to \mathbb{R}$ and $f^{-1} \in (r, \infty) \in \mathcal{F}$ for all $r \in \mathbb{Q}$ then f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$-measurable.

18. Let $\mathcal{G}$ be a nonempty family of continuous real-valued functions defined on $\mathbb{R}$. Assume that for each $x \in \mathbb{R}$ there exists $C_x \in \mathbb{R}$ such that $f(x) \leq C_x$ for all $f \in \mathcal{G}$. Prove that the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \sup\{f(x) \mid f \in \mathcal{G}\}, \quad x \in \mathbb{R},$$

is Borel measurable.