ASSIGNMENT 3: MEASURE THEORY

1. Deduce the monotone convergence theorem from Fatou's lemma.

2. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $\{f_n\}$ is a sequence of nonnegative measurable functions. Show that the inequality

   $$\limsup_{n\to\infty} \int_\Omega f_n \, d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n \, d\mu$$

   is not true in general. (Hint: Try to construct a sequence of functions, all with integral 1 but $\limsup f_n = 0$.)

   What happens if we further assume that $f_n \leq f$ for all $n \in \mathbb{N}$, with f integrable?

3. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f : \Omega \to \mathbb{R}$ is an integrable function such that for all $E \in \mathcal{F}$

   $$\int_E f \, d\mu \geq 0.$$

   Prove that $f(\omega) \geq 0$ for a.e. $\omega \in \Omega$.

4. Prove that the function

   $$f(x) = \begin{cases} \dfrac{1}{x^\alpha}, & x \geq 1, \\ 0, & x < 1, \end{cases}$$

   is integrable on $\mathbb{R}$ if and only if $\alpha > 1$.

   (Hint: You have seen it as an improper Riemann integral. Try to apply MCT.)

5. Prove that the function

   $$f(x) = \begin{cases} \dfrac{1}{x^\alpha}, & 0 < x \leq 1, \\ 0, & x \notin (0,1], \end{cases}$$

   is integrable on $\mathbb{R}$ iff $\alpha < 1$.

6. Suppose $(\Omega, \mathcal{F}, \mu)$ is a finite measure space, that is, $\mu(\Omega) < \infty$. Prove that every bounded measurable function is integrable. Is this true for infinite measure spaces?

7. Determine all $\alpha \in \mathbb{R}$ such that

   $$\int_{(0,\infty)} e^{-x} x^\alpha \, d\lambda(x) < \infty.$$

   (Hint: Decompose the integral suitably. This should remind you of the Gamma function.)

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. If f is integrable then is xf integrable? If xf is integrable then is f integrable?

   (Here $(xf)(x) = xf(x)$.)

   (Hint: Have a look at Problems 4 and 5.)

9. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and xf is also integrable. Prove that the function

$$F(y) = \int_{\mathbb{R}} f(x)\sin(xy)\,dx$$

is differentiable.

(Hint: First find what should be the derivative and then try to use DCT to justify your guess.)

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $f(x) = 0$ for $|x| > 1$. Evaluate

$$\lim_{n\to\infty} \int_{[0,1]} x^n f(x)\,d\lambda(x).$$

(Hint: A typical application of DCT.)

11. Evaluate the sum

$$\sum_{n=0}^{\infty} \left(\int_{[0,\pi/2]} (1 - \sqrt{\sin x})^n \cos x\,d\lambda(x) \right).$$

(Hint: Interchange sum and integral.)

12. Evaluate

$$\lim_{n\to\infty} \int_{(0,1]} \frac{n\cos x}{1 + n^2 x^{3/2}}\,d\lambda(x).$$

13. Fix $0 < a < b$ and define $f_n(x) = ae^{-nax} - be^{-nbx}$, $x \in (0,\infty)$. Prove

   (a) $\displaystyle\sum_{n=1}^{\infty} \int_{[0,\infty)} |f_n|\,d\lambda = \infty.$

   (b) $\displaystyle\sum_{n=1}^{\infty} \int_{[0,\infty)} f_n\,d\lambda = 0.$

   (c) $\displaystyle\int_{[0,\infty)} \sum_{n=1}^{\infty} f_n\,d\lambda$ does not exist.

14. Construct sequences of integrable real-valued functions $\{f_n\}$ and $\{g_n\}$ such that

   (a) $f_n \to 0$ a.e. but $\int_{\mathbb{R}} |f_n|\,d\lambda \not\to 0$.

   (b) $\int_{\mathbb{R}} |g_n|\,d\lambda \to 0$ but $g_n \not\to 0$ a.e.

15. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and $\{f_n\}, f$ are integrable.

   (a) Prove that

$$\int_{\Omega} |f - f_n|\,d\mu \to 0 \quad \Rightarrow \quad \int_{\Omega} |f_n|\,d\mu \to \int_{\Omega} |f|\,d\mu.$$

(b) If $f_n \to f$ a.e., prove that

$$\int_\Omega |f_n|\,d\mu \to \int_\Omega |f|\,d\mu \quad \Rightarrow \quad \int_\Omega |f - f_n|\,d\mu \to 0.$$

(Hint: Apply Fatou's lemma to $g_n = |f| + |f_n| - |f - f_n|$.)

16. Does there exist a nonnegative Lebesgue measurable function f on $\mathbb{R}$ such that for all $E \in \mathcal{L}$
$$\delta_0(E) = \int_E f\,d\lambda?$$

17. Suppose $f : \Omega \to \mathbb{R}$ is integrable. Prove that
$$\sup_{\alpha > 0} \alpha\,\mu(\{\omega \in \Omega : \ |f(\omega)| > \alpha\}) \leq \int_\Omega |f|\,d\mu,$$
where δ_0 is the Dirac delta measure at 0. (This is Chebyshev's inequality.)

    Give an example to show that $\int_\Omega |f|\,d\mu = \infty$, but
$$\sup_{\alpha > 0} \alpha\,\mu(\{\omega \in \Omega : \ |f(\omega)| > \alpha\}) < \infty$$

18. Suppose $(\Omega, \mathcal{F}, \mu)$ is σ-finite. If $f : \Omega \to [0, \infty]$ is integrable then prove there exists a sequence $\{f_n\}$ of integrable functions such that $f_n \uparrow f$, each f_n vanishes outside a set of finite measure, and
$$\lim_{n \to \infty} \int_\Omega f_n\,d\mu = \int_\Omega f\,d\mu.$$

19. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable and continuous at $x_0 \in \mathbb{R}$. Evaluate
$$\lim_{n \to \infty} n \int_{[x_0, x_0 + \frac{1}{n}]} f(x)\,d\lambda(x).$$
(Hint: The answer is $f(x_0)$.)

20. Let $f : \Omega \to (0, \infty)$ be measurable. For each $i \in \mathbb{Z}$ define
$$a_i = \mu\big(f^{-1}((2^{i-1}, 2^i])\big).$$
Prove that f is integrable iff
$$\sum_{i=-\infty}^{\infty} 2^i a_i < \infty,$$
where
$$\sum_{i=-\infty}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=-n}^{n} a_i.$$