ASSIGNMENT 4: MEASURE THEORY

1. If $p \in [1, \infty]$ and $f_n \to f$ in $L^p(\Omega)$ then prove that $\|f_n\|_p \to \|f\|_p$.

2. Suppose $\{f_n\} \subset L^p(\Omega)$, $p \in [1, \infty)$, and $\|f_n\|_p \leq 1$. If $f_n \to f$ pointwise almost everywhere then prove that $f \in L^p(\Omega)$ with $\|f\|_p \leq 1$.

3. If $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ in $L^{p'}(\Omega)$ then prove that $f_n g_n \to fg$ in $L^1(\Omega)$.

4. Suppose $(\Omega, \mathcal{F}, \mu)$ is a finite measure space.
   (a) If $f_n \to f$ in $L^p(\Omega)$ then prove that $f_n \to f$ in $L^q(\Omega)$ for all $q \in [1, p)$.
   (b) If in addition to (a), $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$, then prove that $f_n \to f$ in $L^q(\Omega)$ for all $q \in [1, \infty)$.
   (c) If $\{E_n : n \in \mathbb{N}\} \subset \mathcal{F}$ is such that $\mu(E_n) < \infty$ for all n and $\chi_{E_n} \to f$ in $L^1(\Omega)$ the f is (a.e equal to) the characteristic function of a measurable function.

5. Suppose $1 \leq p < q \leq \infty$ and $f \in L^p(\Omega) \cap L^q(\Omega)$. Prove that $f \in L^r(\Omega)$ for all $r \in (p, q)$.

6. Given $p \in [1, \infty)$, construct $f \in L^p(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that $fg \notin L^p(\mathbb{R})$.

7. For $p \in (1, \infty)$, if $f_n \to f$ in $L^p(\Omega)$ then prove that $|f_n|^p \to |f|^p$ in $L^1(\Omega)$.
   (Hint: $|a^p - b^p| \leq p(|a| + |b|)^{p-1}|a - b|$ and apply Hölder's inequality.)

8. Suppose $f \in L^p(\Omega)$, $g \in L^p(\Omega)$, $p \in (1, \infty)$, with $fg = 0$ almost everywhere. Prove that
$$\|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p.$$

9. If $f \in L^1(\mathbb{R})$ and $h \in L^\infty(\mathbb{R})$ then prove that $\|fh\|_1 = \|f\|_1 \|h\|_\infty$ if and only if $|h(x)| = \|h\|_\infty$ for almost every x such that $f(x) \neq 0$.

10. If $f \in L^\infty([0, 1], \lambda)$ then prove that
$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

11. Given $1 \leq p, q, r < \infty$ with
$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$
prove the following generalization of Hölder's inequality:
$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

12. Given $1 \leq p, q < \infty$ and $\alpha \in [0, 1]$, let $r = \alpha p + (1 - \alpha) q$. Prove Lyapunov's inequality:
$$\|f\|_r^r \leq \|f\|_p^{\alpha p} \|f\|_q^{(1-\alpha)q}.$$

13. For $p \in [1, \infty)$, suppose $\{f_n\} \subset L^p(\Omega)$ and $f \in L^p(\Omega)$ such that $f_n \to f$ pointwise almost everywhere. Prove that $f_n \to f$ in $L^p(\Omega)$ if and only if $\|f_n\|_p \to \|f\|_p$.

14. Prove that $L^p([0, 1], \lambda)$ is separable.

15. Fix $p \in [1, \infty)$. For $f : \mathbb{R} \to \mathbb{C}$, and $x \in \mathbb{R}$, define

$$\tau_x f(y) = f(y - x), \quad y \in \mathbb{R}.$$

   (a) Fix an $f \in C_c(\mathbb{R})$. Prove that the map $F : \mathbb{R}^n \to L^p(\mathbb{R}, \lambda)$, defined by

$$F(x) = \tau_x f, \quad x \in \mathbb{R},$$

   is uniformly continuous.

   (b) Fix an $f \in L^p(\mathbb{R}, \lambda)$. Prove that the map $F : \mathbb{R}^n \to L^p(\mathbb{R}, \lambda)$, defined by

$$F(x) = \tau_x f, \quad x \in \mathbb{R},$$

   is uniformly continuous.

16. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space.

   (a) Define the essential range of a function $f \in L^\infty(\Omega)$ to be the set R_f consisting of all complex numbers z such that

$$\mu(\{\omega \in \Omega : |f(\omega) - z| < \epsilon\}) > 0, \quad \forall\, \epsilon > 0.$$

   Prove that R_f is compact. What is the relation between R_f and $\|f\|_\infty$.

   (b) For $f \in L^\infty(\Omega)$, define

$$A_f = \{\frac{1}{\mu(E)} \int_E f \, d\mu : E \in \mathcal{F},\ \mu(E) > 0\}.$$

      i. What relations exist between A_f and R_f?
      ii. Is A_f always closed?
      iii. Are there measures μ such that A_f is convex for every $f \in L^\infty(\Omega, \mu)$?
      iv. Are there measures μ such that A_f fails to be convex for some $f \in L^\infty(\Omega, \mu)$?