Missed a few lectures.

→ 5 - Mar

→ 9 - Mar

→ 11 - Mar

→ 12 - Mar

Added J's scanned notes right after this page. I apologize.

Order of the notes:

→ J's

→ Mine J sandwich :)

→ J's

5/3/26

- **System of differential Equations:**

$$y' = f(x,y) \quad ; \quad y(x_0) = y_0$$

$$y_1' = f_1(x, y_1, \dots, y_n)$$
$$y_2' = f_2(x, y_1, \dots, y_n)$$
$$\vdots \qquad \vdots$$
$$y_n' = f_n(x, y_1, \dots, y_n)$$

$$\left.\begin{array}{l} y = (y_1, \dots, y_n) \\ f = (f_1, \dots, f_n) \end{array}\right\} \text{n-tuple}$$

$$y' = f(x, y)$$

Suppose $\phi_1, \phi_2, \dots, \phi_n$ exists, then $\phi_1' = f_1(x, \phi_1, \dots, \phi_n)$

Then, $\phi = (\phi_1, \dots, \phi_n)$ is a sol^n of (1).

- **System of Linear DE:**

$$y_1' = a_{11}y_1 + \dots + a_{1n}y_n + b_1(x)$$
$$y_2' = a_{21}y_1 + \dots + a_{2n}y_n + b_2(x)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$y_n' = a_{n1}y_1 + \dots + a_{nn}y_n + b_n(x)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}' = y' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boxed{y' = Ay + B(x)}$$

$B(x) = 0$ homogeneous

$B(x) \neq 0$ non-homogeneous.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Ex: $y_1' = y_1 + 2y_2$
$y_2' = 3y_1 + 4y_2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

→ Suppose we have the n^{th} order d.e, $(y)^{(n)} = f(x, y, \ldots, y^{n-1})$, then $(\phi_1, \phi_1', \ldots, \phi_1^{n-1})$ is the solⁿ.

- **Reduction of n^{th} order eqⁿ to a system of d.e:**

Consider transformation

$y_1 = y$
$y_2 = y'$
$y_3 = y''$
$\vdots$
$y_{n-1} = y^{n-2}$
$y_n = y^{n-1}$

$\Rightarrow$

$y_1' = y' = y_2$
$y_2' = y'' = y_3$
$y_3' = y''' = y_4$
$\vdots$
$y_{n-1}' = y^{(n-1)} = y_n$
$y_n' = y^{(n)}$

- Consider the n^{th} order eqⁿ as $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x)$, $a_0(x) \neq 0$

$$y^{(n)} = (-a_1/a_0) y^{(n-1)} + \cdots + (-a_n/a_0) y + \frac{b(x)}{a_0}.$$

Consider the transformation

$y_1 = y$
$y_2 = y'$
$y_3 = y''$
$\vdots$
$y_{n-1} = y^{(n-2)}$
$y_n = y^{(n-1)}$

$\Rightarrow$

$y_1' = y' = y_2 = 0y_1 + 1y_2 + 0y_3 + \cdots + 0y_n$
$y_2' = y'' = y_3 = 0y_1 + 0y_2 + 1y_3 + \cdots + 0y_n$
$y_3' = y''' = y_4 = 0y_1 + \cdots + 0y_3 + 1y_4 + \cdots + 0y_n$
$\vdots$
$y_{n-1}' = y^{(n-1)} = y_n = 0y_1 + 0y_2 + \cdots + 1y_n$
$y_n' = y^{(n)} = (-a_n/a_0) y_1 + \cdots + (-a_1/a_0) y_n + \frac{b(x)}{a_0}$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n/a_0 & -a_{n-1}/a_0 & -a_{n-2}/a_0 & \cdots & -a_1/a_0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b/a_0 \end{pmatrix}$$

$$\Rightarrow \underline{y' = Ay + B}$$

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Recap? $y^{(n)} = f(x, y, y', \dots, y^{n-1})$

$$y(x_0) = a_0,\ y'(x_0) = a_1, \dots, y^{(n-1)}(x_0) = a_{n-1}$$

$$\phi(x_0) = (a_0, a_1, \dots, a_{n-1})$$

Ex.
(Q) Find the matrix form of following de.,

$$y''' - 4y'' + 10y' - 6y = 9x$$

$y_1 = y \qquad y_1' = y' = y_2$

$y_2 = y' \Rightarrow y_2' = y'' = y_3$

$y_3 = y'' \qquad y_3' = y''' = 4y'' - 10y' + 6y + 9x$

$y_3' = 4y_3 - 10y_2 + 6y + 9x$

$$\Rightarrow \begin{aligned} y_1' &= 0\cdot y_1 + 1\cdot y_2 + 0\cdot y_3 \\ y_2' &= 0\cdot y_1 + 0\cdot y_2 + 1\cdot y_3 \\ y_3' &= 6y_1 - 10y_2 + 4y_3 + 9x \end{aligned} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -10 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 9x \end{pmatrix}$$

$$\Rightarrow y' = Ay + B$$

- Existence and Uniqueness of the system of DE:

Thm: Let $A(x)$ be a continuous $n \times n$ matrix defined on a closed and bdd interval I.

[Existence implied by uniqueness] Then, the IVP,

$$\left.\begin{aligned} y' &= A(x)\,y(x) \\ y(x_0) &= y_0, \quad x, x_0 \in I \end{aligned}\right\} (1)$$

has a unique solⁿ on I.

Pf: First we note that solving (1) is equivalent to solving integral eqⁿ,

$$y(x) = y_0 + \int_{x_0}^{x} A(t)\,y(t)\,dt$$
$$y(x_0) = y_0.$$

Here $\text{norm}(A) := |A| = \Sigma |a_{ij}|$, $A' = (a_{ij}')$

Consider the approximation,

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} A(t)\,y_n(t)\,dt$$

We will prove that $y_n(x)$ cgs unif to $y(x)$.

→ First we claim that $\{y_n\}$ is a Cauchy seq.,

We note that y_n is n^{th} partial sum of the series,

$$y_0 + \Sigma_1^{\infty} (y_{n+1} - y_n)$$

Consider $y_1(x) - y_0 = \int_{x_0}^{x} A(t) y_0(t)\,dt$

$$|y_1 - y_0| \le \int_{x_0}^{x} |A(t)|\,|y_0|\,dt$$

$$\le M|y_0| \int_{x_0}^{x} dt$$

$$= M|y_0|(x - x_0)$$

$$|y_2 - y_1| = \left|\int_{x_0}^{x} A(t)\,(y_1 - y_0)\,dt\right|$$

$$\le M \int_{x_0}^{x} |y_1 - y_0|\,dt$$

$$\le M^2 \frac{|y_0|(x - x_0)^2}{2}$$

$$\therefore\ |y_{n+1} - y_n| \le M^{n+1} \frac{|y_0|(x - x_0)^{n+1}}{(n+1)!} \longrightarrow (2).$$

- RHS of (2) is the $(n-2)^{th}$ term of series $\sum_1^\infty \frac{M^{n-1}|y_0|\,(|y_0|(x-x_0))^{n-1}}{(n-1)!}$ which is a cgt series $|y_0|e^{M(x-x_0)}$.
- Here, $\{y_n\}$ is a cauchy sequence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete, $\{y_n\}$ cgs in norm & norm cgence is unif. Thus, $\{y_n\}$ cgs unif to $y(x)$.

→ **Claim: Sol^n is unique,**

Let y_1 & y_2 be two sol^{ns} of (1).

$$y_1(x) = y_0 + \int_{x_0}^{x} A(t) y_1(t)\,dt$$

$$\& \ y_2(x) = y_0 + \int_{x_0}^{x} A(t) y_2(t)\,dt$$

$$|y_1(x) - y_2(x)| \le M \int_{x_0}^{x} |y_1(t) - y_2(t)|\,dt$$

– By Gronwall's inequality, $y_1 \equiv y_2$

$$|y_1 - y_2| \le M \int_{x_0}^{x} |y_1 - y_2|\,dt \le \varepsilon + M\int_{x_0}^{x} |y_1(t) - y_2(t)|\,dt \text{ for } \varepsilon$$

Let $w(x) = |y_1 - y_2|$

$$\gamma(x) = \varepsilon + M\int_{x_0}^{x} |y_1(t) - y_2(t)|\,dt$$

$$\gamma(x_0) = \varepsilon$$

$$\gamma'(x) = M w(x) \le M\gamma(x) \quad (\because w(x) \le \gamma(x))$$

$$\gamma' - M\gamma \le 0$$

1^{st} order ineq., solving it we get,

$$\gamma(x) e^{-Mx} - \gamma(x_0) e^{-Mx_0} \le 0$$

$$\gamma(x) = \gamma(x_0) e^{M(x - x_0)} \le 0$$

$$w(x) \le 0$$

$\Rightarrow |y_1 - y_2| \le 0$

< 0 not possible

$\Rightarrow |y_1 - y_2| = 0$

$\Rightarrow y_1 = y_2$ sol^n unique

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Recap:

Existence & uniqueness : $\left.\begin{array}{l} y' = A(x)y \\ y(x_0) = y \end{array}\right\}$ ①.

set of sol^n $\{\phi_1, \ldots, \phi_n\}$ of ① form a vector space of dim n.

Let $\phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix}$ .... $\phi_n = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{bmatrix}$

$\phi = (\phi_1, \phi_2, \ldots, \phi_n)$

If $\phi_1, \ldots, \phi_n$ are L.I, then the sol^n is called Fundamental Matrix Soln.

[Fundamental Set of sol^n]

If $\phi_1, \ldots, \phi_n$ are sol^{ns} of $y' = A(x)y$, then Wronskian of sol^{ns} $W(\phi_1, \ldots, \phi_n)$ is equal to $\begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \phi_{21} & - & \phi_{2n} \\ \vdots & & \vdots \\ \phi_{n1} & - & \phi_{nn} \end{vmatrix}$

Ex: Find the fundamental matrix sol^n of $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} x & 0 \\ 0 & 2x \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$y_1' = xy_1$

$y_2' = 2xy_2$

$y_1' = xy_1 \Rightarrow y_1'/y_1 = x \Rightarrow y_1 = c_1 e^{x^2/2}$

$y_2' = 2xy_2 \Rightarrow y_2'/y_2 = 2x \Rightarrow c_2 e^{x^2}$

$y_1 = \begin{pmatrix} e^{x^2/2} \\ 0 \end{pmatrix}$ $y_2 = \begin{pmatrix} 0 \\ e^{x^2} \end{pmatrix}$

$\phi = \begin{bmatrix} e^{x^2/2} & 0 \\ 0 & e^{x^2} \end{bmatrix}$

As y_1, y_2 LI, ϕ_1, ϕ_2 are LI $\Leftrightarrow W|\phi_1, \ldots, \phi| \ne 0$ true.

Ex: $y' = Ay$ $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ $A = \begin{pmatrix} -x & x & 0 \\ 0 & -x & 0 \\ 0 & 0 & x \end{pmatrix}$

$y_2' = -xy_2 \Rightarrow y_2'/y_2 = -x \Rightarrow y_2 = c_2 e^{-x^2/2}$

$y_1' = -xy_1 + xy_2$

$y_3' = xy_3 \Rightarrow y_3'/y_3 = x \Rightarrow y_3 = c_3 e^{x^2}$

$y_1' = -xy_1 + c_2 x e^{-x^2/2}$

$y_1' + xy_1 = xe^{-x^2/2}$

$P = x \Rightarrow A' = P = x$

$\Rightarrow A = x^2/2.$

$$y_1 = e^{-A(x)} \int e^{A(t)} Q(t)\,dt + ce^{-A(x)}$$

$$= e^{-x^2/2} \int e^{t^2/2} te^{-t^2/2}\,dt + ce^{-x^2/2}$$

$$= e^{-x^2/2} x^2/2 + ce^{-x^2/2}$$

$$= e^{-x^2/2}(x^2/2 + c)$$

Eg: After finding M, compute the first two successive approx's of system.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} \sin x & 0 \\ 0 & \cos x \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } I = [0, \pi]$$

$\int \cos t(1+\sin t)$

$\Rightarrow \int u\,du$

$\Rightarrow u^2/2$

$\Rightarrow (1+\sin t)^2/2$

$M \to$ norm bound $\Rightarrow \left\| \begin{matrix} \sin x & 0 \\ 0 & \cos x \end{matrix} \right\| \leq 2$?

$$y_n(x) = y_0 + \int_{x_0}^{x} A(t)\, y_{n-1}(t)\,dt$$

$y_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$y_1 = y_0 + \int_{x_0}^{x} \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = y_0 + \int_{x_0}^{x} \begin{pmatrix} 0 \\ \cos t \end{pmatrix} dt$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ +\sin(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \sin x \end{pmatrix}.$$

$$y_2 = y_0 + \int_{x_0}^{x} \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ 1+\sin t \end{pmatrix} dt = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^x \begin{pmatrix} 0 \\ \cos t(1+\sin t) \end{pmatrix} dt = \begin{pmatrix} 0 \\ 1 + \frac{(1+\sin x)^2}{2} - \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{(1+\sin x)^2}{2} + \frac{1}{2} \end{pmatrix}$$

→ Thm: Let $A(x)$ be an $n \times n$ matrix on I & let ϕ satisfy $y' = A(x)y$. Then, $\det(\phi)$ satisfies the first order eq^n $(\det(\phi))' = tr(A)\det(\phi)$.

Equivalently, $\det[\phi(x)] = \det(\phi(x_0))\, e^{\int_{x_0}^{x} tr(A(t))\,dt}$ ⟶ (Abel's formula)

Proof: Given that ϕ satisfies the eq^n $y' = Ay$

$\Rightarrow \phi' = A\phi$

$\Rightarrow \phi_{ij}' = \sum a_{ik}\phi_{kj} \quad i,j = 1,\dots,n$

Let $\phi = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \phi_{21} & \cdots & \phi_{2n} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{pmatrix}$, $\det(\phi) = \begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \phi_{21} & \cdots & \phi_{2n} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{vmatrix}$

$$(\det(\phi))' = \begin{vmatrix} \phi_{11}' & \phi_{12}' & \cdots & \phi_{1n}' \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & & \vdots \\ \phi_{n1}' & \phi_{n2}' & \cdots & \phi_{nn}' \end{vmatrix}$$

Consider first term, $\begin{vmatrix} \phi_{11}' & \phi_{12}' & \cdots & \phi_{1n}' \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = \begin{vmatrix} \sum a_{1k}\phi_{k1} & \sum a_{1k}\phi_{k2} & \cdots & \sum a_{1k}\phi_{kn} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}\phi_{11} & a_{11}\phi_{12} & \cdots & a_{11}\phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} + 0.$

$= a_{11} \det\phi.$

Continuing ||ly,

$$(\det(\phi))' = (a_{11} + a_{22} + \cdots + a_{nn}) \det(\phi) = tr(A) \det\phi.$$

$$\Rightarrow \frac{(\det(\phi))'}{\det\phi} = tr(A) \Rightarrow \det\phi(x) = \det(\phi(x_0))\, e^{\int_{x_0}^{x} tr(A(t))\,dt}.$$

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Thm: A solution matrix ϕ of the matrix diff eq^n $y' = A(x)y$, $x \in I$. is a fundamental Matrix sol^n $\Leftrightarrow$ $\det(\phi) \neq 0$ for any $x \in I$. $\longrightarrow$ ①

Pf: ($\Rightarrow$) Suppose ϕ be a FMS. Then, WKT the components of ϕ, namely $\phi_1, \phi_2, \ldots, \phi_n$ forms a basis of the solution space.

Let ψ be any solution of ①. Since $\{\phi_i\}$ is a basis,

then $\exists!$ c_i's s.t $\psi(x) = \sum c_i \phi_i(x)$

$\Rightarrow \psi_i(x) = \sum \phi_{ij} c_j$, $i = 1, \ldots, n$.

This is a sys of n linear eq^n.

Since this sys has non-trivial solution for any fixed $x_0 \in I$, we have

$\det(\phi(x_0)) \neq 0$

$\Rightarrow \det(\phi(x)) \neq 0$ (from last thm).

($\Leftarrow$) Conversely, suppose $\det(\phi(x)) \neq 0$

$\Rightarrow W(\phi_1, \ldots, \phi_n)(x) \neq 0$

$\Rightarrow \phi_1, \ldots, \phi_n$ are L.I

$\Rightarrow \phi = (\phi_1, \ldots, \phi_n)$ is a FMS (Pd)

Thm: (i) If ϕ is a FMS of $y' = A(x)y$ & C is any constt non-singular matrix, then ϕC is a FMS

(ii) Every FM of the system is of this type for some nonsingular matrix

Pf: (i) Given that ϕ is FMS of $y' = A(x)y$

$\Rightarrow \det(\phi) \neq 0$ (prev thm)

$\Rightarrow \phi$ is non singular.

WKT, c is also nonsingular

$\Rightarrow \phi c$ is nonsingular.

$\Rightarrow \det(\phi c) \neq 0$

$\Rightarrow \phi C$ is FMS (Pd)

(ii) Let ϕ_1 and ϕ_2 be the FM of $y' = A(x)y$.
Let $\phi_2 = \phi_1 \psi$.

We claim that ψ is non-singular constt matrix.

$$\phi_2 = \phi_1 \psi$$
$$\Rightarrow \phi_2' = \phi_1' \psi + \phi_1 \psi'$$
$$\Rightarrow A\phi_2 = A\phi_1 \psi + \phi_1 \psi'$$
$$= A\phi_2 + \phi_1 \psi'$$
$$\Rightarrow \phi_1 \psi' = 0 \Rightarrow \psi' = 0$$
$$\Rightarrow \psi = C$$
$$\therefore \phi_2 = \phi_1 C \Rightarrow C = \phi_1^{-1}\phi_2$$ (Pd)

Thm: Let A be a constt matrix and ϕ be the FM of the sys $y' = Ay$ st $\phi(0) = E$, the identity matrix. Then, ϕ satisfies $\boxed{\phi(x+t) = \phi(x)\phi(t) \quad \forall x, t \in I}$

Pf: Let us denote $Y(x) = \phi(x+t)$
$Z(x) = \phi(x)\phi(t)$

TPT $Y(x) = Z(x)$

We claim that Y & Z are solns of $y' = Ay$ with same initial condn.

Consider $Y'(x) = \phi'(x+t) = A\phi(x+t) = AY(x)$,
$Y(0) = \phi(t)$

Consider $Z'(x) = \phi'(x)\phi(t) = A\phi(x)\phi(t) = AZ(x)$
$Z(0) = \phi(0)\phi(t) = E\phi(t) = \phi(t)$.

Then, Y and Z are solns of $y' = Ay$ with same initial condn $Y(0) = Z(0) = \phi(t)$.

Then by Existence & Uniqueness thms, $Y = Z$
$$\Rightarrow \phi(x+t) = \phi(x)\phi(t)$$ (Pd)

Solution of System of Non-homogeneous Eqns

We have $y' = A(x)y + B(x)$ $\quad B(x) \neq 0$

Thm: Let ϕ_0 be any soln of non-homogeneous sys $y' = A(x)y + B(x) \to (1)$ and $\phi_1, \phi_2, \ldots, \phi_n$ be a basis of solns of the hom. sys $y' = A(x)y \to (2)$ and let $c_1, \ldots, c_n$ be constts. Then,

(i) the vector function $\phi_0 + \sum_{i=1}^{n} c_i \phi_i$ is also a soln of sys (1) for every choice of c_i's. $\to (3)$

(ii) Any arbitrary solⁿ of (1) is of form (3) for suitable choice of c_i's.

Pf: Given that ϕ_0 is a solⁿ of the non-homo. syst $y' = A(x)y + B(x)$ (1)

$\Rightarrow \phi_0' = A\phi_0 + B$

Also, we have $\phi_1, \ldots, \phi_n$ is a basis of solⁿ of $y' = Ay$.

$\Rightarrow \sum c_i\phi_i$ is also a solⁿ of $y' = A(x)y$

$\Rightarrow (\sum c_i\phi_i)' = A\sum c_i\phi_i$.

Consider $(\phi_0 + \sum c_i\phi_i)' = \phi_0' + (\sum c_i\phi_i)'$

$= A\phi_0 + B + A\sum c_i\phi_i$

$= A(\phi_0 + \sum c_i\phi_i) + B$

$\Rightarrow \phi_0 + \sum c_i\phi_i$ is a solⁿ of (1) ~~nd~~

(ii) Let ϕ be any solⁿ of (1)

Claim that $\phi - \phi_0$ is a solⁿ of (2),

Consider $(\phi - \phi_0)' = \phi' - \phi_0'$

$= A\phi + B - (A\phi_0 + B)$

$= A(\phi - \phi_0)$

$\Rightarrow \phi - \phi_0$ is a solⁿ of (2)

Since $\phi_1, \ldots, \phi_n$ is a basis of ~~solⁿ~~ space of (2)

$\phi - \phi_0 = \sum c_i\phi_i$ for some c_i

$\Rightarrow \phi = \phi_0 + \sum c_i\phi_i$ Pd.

Lect - 4, Mar 16

Particular soln of a non-homog. system using variation of parameter technique

Theorem:

If $\phi(x)$ is the FM of the homogenous system $y' = A(x).y$, $x \in I$, Then ψ defined by

$$\psi(x) = \phi(x) \int_{x_0}^{x} \phi^{-1}(t).\ B(t).\ dt$$

is a solution of IVP of the non-homogenous system, $y' = A(x)y + B(x)$, $y(x_0) = 0$

Proof:

Given That ϕ is a FM of

$y' = A(x).\ y$ ——— ①

Let us assume that,

$\psi = \phi u$

is a solution of $y' = Ay + B$ ——— ②

ψ is a solution of ②.

$\Rightarrow \quad \psi' = A\psi + B$

$\psi = A\phi u + B$ ——— ③

But $\psi' = (\phi u)'$

$= \phi' u + \phi u'$ ——— ④

From ③ and ④,

$Au\phi + B = A\phi u + \phi u'$

$\Rightarrow \quad B = \phi u'$

$\Rightarrow \quad u' = \phi^{-1}.\ B$

$\Rightarrow \quad u(x) = \int_{x_0}^{x} \phi^{-1}(t).\ B(t).\ dt$

$\Rightarrow \quad \psi(t) = \phi(x).\ \int_{x_0}^{x} \phi^{-1}(t).\ B(t).\ dt$ ——— ⑤

On the other hand, if ψ is of the form ⑤, then ψ satisfies the equation ②.

Question:

Obtain the soln $\psi(x)$ of the IVP,

$y' = Ay + B$ where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \sin ax \\ \cos bx \end{bmatrix}$$

$$\psi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

Step 1: Find the FM ϕ.

Consider,

$$y' = Ay$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\Rightarrow \quad y_1' = y_1, \quad y_1 = e^{x}$

$y_2' = 2y_2, \quad y_2 = e^{2x}$

$\therefore$ the FM,

$$\phi = \begin{bmatrix} e^{x} & 0 \\ 0 & e^{2x} \end{bmatrix}$$

Step 2:

to find the solution ψ, first we compute ϕ^{-1}.

$$\phi^{-1} = \begin{bmatrix} e^{-x} & 0 \\ 0 & e^{-2x} \end{bmatrix}$$

We have,

$$\psi = \phi . \int_{x_0}^{x} \phi^{-1}(t).\ B(t).\ dt$$

$$= \phi \int_{x_0}^{x} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} \sin at \\ \cos bt \end{bmatrix} dt$$

$$= \phi \int_{0}^{x} \begin{bmatrix} e^{-t}.\ \sin at \\ e^{-2t}.\ \cos bt \end{bmatrix} . dt$$

To note:

$$\int e^{ax}.\cos bx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C_1$$

$$\int e^{ax}.\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C_2$$

$$\phi(x) = \begin{bmatrix} e^{x} & 0 \\ 0 & e^{2x} \end{bmatrix} \begin{bmatrix} \frac{e^{-x}}{1+a^2}(-\sin ax - a\cos ax) + C_1 \\ \frac{e^{-2x}}{4+a^2}(-2\cos ax - a\sin ax) + C_2 \end{bmatrix}$$

$$\phi(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow C_1 = \frac{a}{1+a^2}, \quad C_2 = \frac{6+b^2}{4+b^2}$$

$$\varphi(x) = \begin{bmatrix} \frac{1}{1+a^2}(ae^{x} - (\sin ax - a\cos ax)) \\ \frac{1}{4+b^2}((6+b^2)e^{2x} + b\sin bx - 2\cos bx) \end{bmatrix}$$

Linear system with constant coefficients

Theorem:

The general solution of $y' = Ay$ ——— ①

where A is a constant matrix, is

$$y(x) = e^{xA}.C \quad \text{where } C \text{ is}$$

arbitrary constant column vector.

② If ① satisfies the initial condition $y(x_0) = y_0$ then the solution is $y(x) = e^{(x-x_0)A}.y_0$.

③. The FM of ① is $\phi(x) = e^{xA}$.

Proof:

Consider $y' = Ay$ ——— ①

$\Rightarrow \quad 0 = y' - Ay$ ——— ②

Let $u(x) = e^{-xA}.y(x)$.

Then,

$$u'(x) = -Ae^{-xA}.y + e^{-xA}.y'$$
$$= (y' - Ay).e^{-xA}$$

$\therefore \quad u' = 0$

$\Rightarrow \quad u = c$

$\Rightarrow \quad e^{-xA}y = c$

$y = c.e^{xA}$ ——— ②

② We have $y(x) = e^{xA}c$

$\Rightarrow \quad y(x_0) = e^{x_0A}c$

$y_0 = e^{x_0A}c$

$c = e^{-x_0A}.y_0$

$\Rightarrow \quad y(x) = e^{xA}.e^{-x_0A}.y_0$

$y(x) = e^{A(x-x_0)}.y_0$

③ Let $\phi = e^{xA}$

$\therefore \quad \phi' = Ae^{xA} = A\phi$

$\Rightarrow \quad \phi$ is a soln of $y' = Ay$.

Consider

trace (A)

$$\det(\phi) = \det(\phi(x_0))\, e^{\int_{x_0}^{x} tr(A).dt}$$
$$= \det(e^{x_0A})\, e^{tr(A).(x-x_0)} \neq 0$$

$\Rightarrow \quad \phi$ is a FM.

Note:

$$e^{xA} = I + xA + \frac{x^2}{2!}A^2 + \ldots\ldots$$

For eq,

suppose $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$

$\therefore$

$$e^{xA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} + \frac{x^2}{2}\begin{bmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + xa_1 + \frac{x^2}{2}a_1^2 + \ldots & 0 \\ 0 & 1 + xa_2 + \frac{x^2}{2}a_2^2 + \ldots \end{bmatrix}$$

lect -5, Mar - 18

Eigenvalue or Eigen vector method to compute 'n' independent solns.

Theorem :

consider a vector differential eqn.

$$y' = Ay(x), \quad x \in I \quad \text{———} \; (1)$$

where A is an nxn constant matrix.

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are real distinct eigenvalue of A and $v_1, v_2, \ldots, v_n$ are the corresponding eigenvectors of A, then $e^{\lambda_1 x} v_1, e^{\lambda_2 x} v_2, \ldots, e^{\lambda_n x} v_n$ for LI soln of (1) and the gen. soln of (1) is

$$y(x) = \sum c_i e^{\lambda_i x} . v_i \quad , \quad c_i\text{'s are contn.}$$

(ii) If $\lambda_1, \lambda_2, \ldots, \lambda_m$, $m < n$ are eigenvalues of A with multiplicity $n_1, n_2, \ldots, n_m$, where

$$n_1 + n_2 + \ldots + n_m = n$$

with the corresponding eigenvalue, then the soln is given by,

$$y(x) = e^{xA} \sum c_i$$

$$= \sum e^{\lambda_i x} \left[\sum_{j=0}^{n_i - 1} \frac{x^j}{j!} (A - \lambda_j E)^j \right] c_i$$

where $y(0) = c = (c_1, c_2, \ldots, c_n)$

Proof.

Given that $y' = Ay$, $A_{n \times n}$ matrix.

$$\text{———} \; (1)$$

let us assume that $y(x) = e^{\lambda x} v$ is a soln of (1). v is a non-zero constant vector.

Then, $y' = \lambda e^{\lambda x} . v$

$$\therefore \; y' = Ay \Rightarrow \lambda e^{\lambda x} . v = A e^{\lambda x} . v$$

$$\Rightarrow (A - \lambda E) e^{\lambda x} . v = 0$$

$$(A - \lambda E) v = 0$$

$$|A - \lambda E| = 0 \quad \text{———} \; (2)$$

Then the charac. eq and has n roots.

Case 1 :

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct roots of (2), then we have n LI eigenvalues, $v_1, \ldots v_n$ and have,

$e^{\lambda_1 x} v_1, \ldots, e^{\lambda_n x} v_n$ are n LI soln of the system (1).

Thus the general soln is $y(x) = \sum c_i e^{\lambda_i x} v_i$

Case 2 :

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalue of (2) with multiplied $n_1, n_2, \ldots, n_m$ such that $n = n_1 + n_2 + \ldots + n_m$.

If λ_i is an eigen value with multiplicity n_i. then $(A - \lambda_i E)^{n_i} v = 0$

Let $x_i \subseteq \mathbb{R}^n$ be the subspace spanned by the eigenvector v_i.

Then we have,

$$\mathbb{R}^n = x_1 \oplus x_2 \oplus + \ldots \oplus x_m$$

Any $c \in \mathbb{R}^n$, $c = c_1 + c_2 + \ldots + c_m$, $c_i \in x_i$.

we know that

$$y(x) = e^{xA} . c \text{ is a soln of } (1), \; c \in \mathbb{R}^n$$

$$= e^{xA} \sum c_i$$

$$= \sum_{i=1}^{m} e^{xA} c_i$$

$$e^{xA} = e^{\lambda_i x} e^{(A - \lambda_i E)x}$$

$$e^{(A - \lambda_i E)x} = E + x(A - \lambda_i E) + \frac{x^2}{2!} (A - \lambda_i E)^2 + \ldots$$

$$e^{xA} c_i = e^{\lambda_i x} \left[E + x(A - \lambda_i E) + \ldots \right] c_i$$

But we know that

$$(A-\lambda_i E)^i v = 0, \quad i \geq n_i$$

$$e^{xA} \cdot c_i = e^{\lambda_i x} \left[\sum_{j=0}^{n_i - 1} \frac{x^j}{j!} (A-\lambda_j E)^j \right] c_i$$

$$\Rightarrow y(x) = \sum_{i=1}^{m} e^{xA} \cdot c_i$$

$$= \sum_{i=1}^{m} e^{\lambda_i x} \left[\cdots\cdots \right] c_i$$

Algorithm for computing the soln of $y' = Ay$.

Step 1 :

Compute the eigenvalue of A.

Step 2 :

Case 1 : If λ_i's are distinct, then $e^{\lambda_i x} v_i$, $i = 1, \ldots, n$ where v_i's are the eigen vectors, are the soln of $y' = Ay$.

Case 2 : Suppose the eigenvalue have multiplicity, For example, multiplicity 2.

The second LI soln is $e^{\lambda x}\left[v + (A-\lambda E)v\right]$

$$(A-\lambda E)^2 v = 0$$

but, $(A-\lambda E) v \neq 0$

This process can be repeated for all other eigen values.

Problem :

$$y' = Ay$$

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Solution :

Step 1 : To find the eigen value.

Consider

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\Rightarrow \quad \lambda = 1, 6.$$

Step 2 :

To find the eigen vectors,

when $\lambda = 1$

Consider $(A - \lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_1 + v_2 = 0 \quad = (1, -1)$$

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

When $\lambda = 6$,

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = 4, \quad v_2 = 1$$

$$v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Problems :

1. $y' = Ay$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Determine e^{xA} and FM of $y' = Ay$.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

Lect - 6, Mar 19

Linear system with periodic coefficients

We say that solution $y(x)$ of $y' = A(x).y$ is periodic with period ω, if $y(x+\omega) = y(x)$.

If the matrix $A(x)$ is said to be of period ω, if $A(x+\omega) = A(x)$.

Theorem :

The system $y' = A(x).y$ admits non-zero periodic solution of period ω.

$\Leftrightarrow$ $E.e^{A\omega}$ is singular,

when E is default matrix.

Proof :

We know that the solution of $y' = A(x).y$ is $y(x) = e^{xA}.c$, $c \neq 0$.

y is periodic iff $y(x+\omega) = y(x)$

$$\Leftrightarrow e^{(x+\omega)A}.c = e^{xA}.c$$

$$\Leftrightarrow e^{xA}.e^{\omega A}.c = e^{xA}.c$$

$$\underbrace{e^{xA}(E - e^{\omega A}).c}_{=0} = 0$$

$$\det(E - e^{\omega A}) = 0$$

$E - e^{\omega A}$ is singular.

Theorem :

Let the matrix $A(x)$ and the function $B(x)$ be continuous and periodic with $\omega \in \mathbb{R}$. Then the differential system,

$$y' = A(x).y + B(x) \quad \text{———} \quad (1)$$

has a periodic solution.

$y(x)$ of period ω $\Longleftrightarrow$ $y(0) = y(\omega)$

Proof :

Suppose (1) has a periodic solution $y(x)$

$\Rightarrow$ $y(x+\omega) = y(x)$

$\Rightarrow$ $y(\omega) = y(0)$.

Conversely suppose (1) has a soln $y(x)$ s.t. $y(0) = y(\omega)$. Then we need to prove that y is periodic, i.e. to prove that,

$$y(x+\omega) = y(x).$$

Let $v(x) = y(x+\omega)$

Claim v is a soln.

$$v'(x) = y'(x+\omega) = A(x+\omega).y(x+\omega) + B(x+\omega)$$

$$= A(x).v(x) + B(x)$$

$\Rightarrow$ v is a soln.

and $v(0) = y(\omega) = y(0)$

$\Rightarrow$ $v(x) = y(x)$ $\{ \because$ uniqueness of soln

$\Rightarrow$ $y(x+\omega) = y(x)$

$\Leftrightarrow$ $(E - e^{\omega A})c = 0$, thus have a non-trivial soln,

$\det(E - e^{\omega A}) = 0$

$\Rightarrow$ $E - e^{\omega A}$ is singular.

Theorem :

Let the matrix $A(x)$ be continuous and periodic with period ω. Further let $\phi(x)$ be a FM of the differential system,

$$y' = A(x).y \quad \text{———} \quad (1)$$

Then (1) has a non-trivial periodic solution $y(x)$ of period ω $\Longleftrightarrow$ $\det(\phi(0) - \phi(\omega)) = 0$

Proof :

Given that $y' = A(x).y$ and ϕ is a FM of (1). ——(1)

We know that $y(x) = \phi(x).c$ is also a soln of ①.

We know that y is periodic

$\iff y(0) = y(\omega)$

$\iff \phi(0).c = \phi(\omega).c$

$\iff (\phi(0) - \phi(\omega)).c = 0$

$\iff \det(\phi(0) - \phi(\omega)) = 0$ for non-trivial ω.

Theorem :

Let the matrix $A(x)$ be continuous and periodic with period $\omega \in \mathbb{R}$. Let $\phi(x)$ be the FM of the system

$$y' = A(x)\,y + B(x) \quad \text{———— ①}$$

Then ① has a unique periodic solution of period ω,

$\iff$ the system $y' = A(x).y$ does not have a periodic soln of period ω.

Proof

Given that ϕ is a FM of $y' = A(x).y$.

Let $x_0 = 0$. We have the solution of ① as

$$y(x) = c.\phi(x) + \phi(t).\int_0^x \phi^{-1}(t).B(t).dt,$$

c is some coefficient.

y is periodic $\iff y(0) = y(\omega)$.

$$\Rightarrow c.\phi(0) = c.\phi(\omega) + \phi(\omega)\int_0^\omega \phi^{-1}(t).B(t).dt$$

$$\Rightarrow (\phi(0) - \phi(\omega))c = \phi(\omega).\int_0^\omega \phi^{-1}(t).B(t).dt$$

This has a non-trivial soln

$\iff \det(\phi(0) - \phi(\omega)) \neq 0.$

$\iff y' = A(x).y$ does not have a periodic soln.

Floquet's theorem

Let the matrix $A(x)$ be continuous and periodic with period $\omega \in \mathbb{R}$. Then the following hold.

(i) The matrix $\psi(x) = \phi(x+\omega)$ is also a FM of the system, $y' = A(x).y$.

(ii) There exists a periodic non-singular matrix $P(x)$ of period ω and a constant matrix R such that

$$\phi(x) = P(x).e^{Rx}$$

Result :

Let A be a non-singular matrix. Then there exist a matrix B such that $A = e^B$

Lect - 7, Mar 23 Boom's birthday !

Floquet's theorem

1. $\phi(x+\omega)$ is a FM
2. $\phi = p.e^{Rx}$, P is periodic and R is cont.

Proof :

(i) Given that ϕ is a FM of $y' = Ay$.

to prove that $\psi(x) = \phi(x+\omega)$ is a FM.

Consider $\psi'(x) = \phi'(x+\omega) = A(x+\omega).\phi(x+\omega)$

$= A(x).\phi(x)$

$\Rightarrow \psi$ is a soln of $y' = A(x).y$.

$\det(\psi(x)) = \det(\phi(x+\omega)) \neq 0$

$\Rightarrow \psi$ is a FM of $y' = Ay$

(ii) We know that $\phi(x)$ and $\phi(x+\omega)$ are FM of $y' = Ay$.

We also know that any soln. can be of the form $\phi(x).C$, where C is non-singular, i.e.,

$$\phi(x+\omega) = C.\phi(x)$$

Let $P(x) = \phi(x).e^{-Rx}$

Since C is non-singular, $C = e^{R\omega}$.

Claim: P is periodic

to prove that, $P(x+\omega) = P(x)$.

Consider, $P(x+\omega) = \phi(x+\omega).e^{-R(x+\omega)}$

$= \phi(x).C.e^{-Rx}.e^{-R\omega}$

$= \phi(x).e^{R\omega}.e^{-Rx}.e^{-R\omega}$

$= \phi(x).e^{-Rx}$

$= P(x)$.

$\Rightarrow$ P is periodic.

$$P = \phi.e^{-Rx}$$

$$\Rightarrow \phi = P.e^{Rx}.$$

Reduction of a periodic system to a system with constant coefficient.

Theorem:

Let $P(x)$ and R be the matrices as in Floquet's theorem. Then the transformation,

$$y(x) = P(x).v(x)$$

reduces the differential system $y' = A(x).y$. to $v' = Rv$ with constant coeff. matrix R.

Proof:

We know that ϕ is a FM of

$$y' = A(x).y \qquad \text{——— } ①$$

By Floquet's theorem we have,

$$\phi = P.e^{Rx},$$

ϕ is a solution of ① $\Rightarrow$ $\phi' = A.\phi$.

$\Rightarrow (Pe^{Rx})' = A(P.e^{Rx})$

$\Rightarrow P'e^{Rx} + Pe^{Rx}.R = APe^{Rx}$

$\Rightarrow e^{Rx}(P' + PR - AP) = 0$

$\Rightarrow P' + PR - AP = 0$

$P' - AP = -PR \qquad \text{——— } ②$

Consider $y = Pv$.

$\therefore y' = Ay$.

$\Rightarrow (Pv)' = A(Pv)$

$\Rightarrow P'v + Pv' = AP.v$

$\Rightarrow (P' - AP)v + Pv' = 0$

$\Rightarrow -PR.v + Pv' = 0$ {from ②}

$\Rightarrow v' - Rv = 0$ {P is non-singular}

$\Rightarrow v' = Rv$

Asymptotic behaviour of linear systems

Theorem:

Let all the solution of the differential system, $y' = Ay$, $y(x_0) = y_0$, A is a constant matrix be bounded. Then all the solutions of perturbed system,

$$y' = (A + B(x)).y,$$

$B(x)$ is an $n\times n$ matrix continuous elements by $1 \le i, j \le n$ in the interval $[x_0, \infty)$, will be bounded, provided

$$\int_{x_0}^{\infty} \|B(t)\|.dt < \infty.$$

Proof :

we have $y' = Ay$ is constant. $y(x_0) = y_0$. —— ①

we know that solution of ① is of the form e^{Ax}. It is given that soln. of ① is bounded.

$\Rightarrow$ $\exists$ some constant c, such that

$$\| e^{Ax} \| \le c$$

we have the perturbed system,

$$y' = (A + B(x))\, y \qquad —— ②$$

The solution of ② is,

$$y(x) = e^{(x-x_0)A} \cdot y_0 + e^{xA} \int_{x_0}^{x} e^{-At} \cdot B(t)\, y(t) \cdot dt$$

$$= e^{(x-x_0)} \cdot y_0 + \int_{x_0}^{x} e^{(x-t)A} \cdot B(t) \cdot y(t) \cdot dt$$

$$\| y(x) \| \le c_0 + c \int_{x_0}^{x} \| B(x) \| \cdot |y(t)| \cdot dt$$

By Grönwall inequality,

$$|y(x)| \le c_0\, e^{\int_{x_0}^{x} \|B(t)\| \cdot dt}$$

$$\le K. \quad \because \int_{x_0}^{x} \|B(t)\| \cdot dt < \infty$$

Theorem :

Let all the soln of the differential system $y' = A(x) \cdot y$, $y(x_0) = y_0$ be bounded in $[x_0, \infty)$ and the condition, $\int \| B(t) \| \cdot dt < \infty$ be satisfied, then all the soln of the perturbed system,

$$y' = (A(x) + B(x)) \cdot y.$$

are bounded in $[x_0, \infty)$, provided

$$\lim_{n \to \infty} \inf \int_{x_0}^{x} Tr(A(t)) \cdot dt > -\infty$$

$$\text{or}, \quad Tr(A(x)) = 0.$$

Proof :

we know that ϕ is the FM soln of $y' = A(x)\, y$.

$$\phi^{-1} = \frac{Adj(\phi)}{\det(\phi)}$$

$$= \frac{Adj(\phi)}{\det(\phi) \cdot \exp\left(\int_{x_0}^{x} tr(A(t))\, dt \right)}$$

Then both ϕ and ϕ^{-1} are bounded { since soln is bdd and due to given condt. }

Let,

$$c = \max \left\{ \sup \|\phi\|, \sup \|\phi^{-1}\| \right\}$$

we have the perturbed system,

$$y' = (A(x) + B(x))\, y \qquad —— ②$$

$$y(x) = \phi(x) \cdot c + \phi \int_{x_0}^{x} \phi^{-1} B(t) \cdot y(t) \cdot dt$$

$$= \phi(x) \cdot \phi(x_0)^{-1} \cdot y_0 + \int_{x_0}^{x} \phi(t) \cdot \phi^{-1}(t) \cdot B(t) \cdot dt$$

$$|y(x)| \le c_0 + c^2 \int_{x_0}^{x} \|B(t)\| \cdot |y(t)| \cdot dt$$

By GI,

$$\Rightarrow |y| \le c_0 \cdot \exp\left\{ c^2 \cdot \int \| B(t) \| \cdot dt \right\}$$

$$\le K.$$

Missed a few lectures:

→ 25 - Mar

→ 26 - Mar

Attached J's at the end. Sorry. You can ignore my notes from here and scroll to her's at the end.

Lect - n, Mar 20

Proposition:

Let all the roots of the characteristic eqn of

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

where a_i's are some constants, have negative real parts and let $e > 0$ be such that $-e > \max_{1 \le j \le m} \text{real} (e^{\lambda_j})$ where

$\lambda_1, \lambda_2, \dots, \lambda_m$ are distinct roots. Then there exists a constant k such that

$$|y(x)| \le k e^{-\delta x}, \quad x \ge 0.$$

Proof:

We have $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$ —— ①

Then, ch. eq of ① is,

$$\lambda^n + a_1 \lambda^{n-1} + \dots + a_n \lambda = 0.$$

Suppose $\lambda_1, \lambda_2, \dots, \lambda_m$ be the roots with multiplicity $q_1, \dots, q_m$ s.t. $q_1 + q_2 + \dots + q_m = n$.

Let $\lambda_j = \alpha_j + i.\beta_j$, $j = 1, \dots, m$. $\alpha_j < 0$.

Let $y_i(x) = x^p e^{\lambda_j x}$, $0 \le p < q_j$, $j = 1, \dots, m$.

consider

$$y_j(x) e^{\delta x} = x^p e^{\lambda_j x} e^{\delta x} = x^p e^{(\lambda_j + i\beta_j)x} e^{\delta x}$$

Note: $\alpha_j + \delta < 0$

$$|y_j(x). e^{\delta x}| = |x^p . e^{(\alpha_j + \delta)x}|$$

$\exists$ a k_j such that

$$|y_j(x). e^{\delta x}| \le k_j$$

$$\Rightarrow \quad |y_j(x)| \le k_j . e^{\delta x}$$

We know that $y(x) = \Sigma c_i y_i(x)$

$$|y(x)| = \Sigma c_i |y_i(x)|$$

$$\le \max c_i \Sigma |y_i(x)|$$

$$|y(x)| \le \max c_i \Sigma k_j e^{-\delta x}$$

$$\le k e^{-\delta x}$$

Theorem:

1. Every solution $y = \phi(x)$ of $y' = Ay$ is stable of all the eigenvalue of A have neg. real part.
2. Every solution $y = \phi(x)$ of $y' = Ay$ is unstable if atleast one eigenvalue of A has the real part.

Proof:

we have $y' = Ay$ —— ①

we know that any solution $\psi(x)$ of ① is

$$\psi(x) = e^{xA} \psi(0).$$

Let ϕ_{ij} be the i-j^{th} component of e^{xA}.

Then,

$$\psi_i(x) = \sum_{j=1}^{n} \phi_{ij} \psi_j(0), \quad i = 1, \dots, n.$$

Let us assume that the eigen values of A has negative real part. Let $-\alpha_i$ be the largest negative real part of such eigen value.

Then, $\exists$ e such that $-\alpha_1 < -\delta < 0$

By the preims (??) proposition, $\exists$ $k > 0$ s.t.

$$|\phi_{ij}(x)| \le k e^{-\delta x}$$

Define $\|\psi(x)\| = \max \{ |\psi_j(x)|, \ j = 1, \dots, n \}$

$$\|\psi(x)\| = \max |\psi_i(x)|$$

$$= \max \left| \sum_{j=1}^{n} \phi_{ij}(x). \psi_j(0) \right|$$

$$\le \max \left| \sum_{j=1}^{n} k.e^{-\delta x} . \psi_j(0) \right|$$

$$= nk \|\psi(0)\|$$

$$\le nk\delta, \quad \|\psi(0)\| < \delta$$

$$< \varepsilon, \quad \text{if chose } \delta = \frac{\varepsilon}{nk}$$

i.e., $\|\psi(x)\| < \varepsilon$ when $\|\psi(0)\| < \delta$.

Any solution $\psi(x)$ sufficiently close to the equilibrium part at $x = 0$ remains close to it infinite times as well.

$\Rightarrow$ the soln is stable.

Proof:

Let λ be any positive eigen value of A.

Then

$$\psi(x) = ce^{\lambda x}.v$$

when v is the eigenvalue.

$$\|\psi(x)\| = \|ce^{\lambda x}.v\| \longrightarrow \infty, \quad x \to \infty.$$

$\Rightarrow$ the soln is unstable.

① $y' = Ay$ when $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$ is unstable.

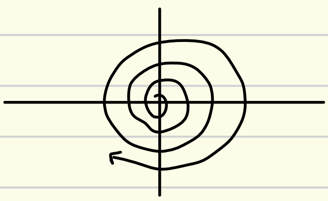
② Determine whether each solution $y(x)$ of $y' = Ay$ when $A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{bmatrix}$

Lect - n+1, Apr - 1

Limit Cycle :

$$\left.\begin{aligned} y_1' &= f_1(x, y) \\ y_2' &= f_2(x, y) \end{aligned}\right\} \text{— ①}$$

A closed trajectory of the differential syst. ①, which is approached specifically either from inside or from outside by a non-closed trajectory of ① as $x \to \infty$ or $x \to -\infty$ is called a Limit cycle.



Poincare - Benedixson Theorem

Suppose that a solution $y(x) = (y_1(x), y_2(x))$ of the differential system $\underbrace{y_1' = f_1(x, y),\ y_2' = f_2(x, y)}_{\to ①}$ remains in a bounded region of y_1- y_2 plane which contains no critical or equilibrium points of ①, then its trajectory must spiral into a simple closed curve which itself is the trajectory of a periodic solution.

Lecture incomplete.

Missed lectures:

→ 2 - Apr

→ 6 - Apr

Added J's handwritten notes after this. Sorry!

25/3/26

Stability of the solution:

$$\left.\begin{array}{l} y' = f(x,y) \\ y(x_0) = y_0 \;,\quad \tilde{y}(x_0) = y_0 + \Delta y_0 \end{array}\right\} ①$$

Let $y(x) = y(x, x_0, y_0)$ be the solⁿ wrt i.c $y(x_0) = y_0$

$\tilde{y} = y(x, x_0, y_0 + \Delta y_0)$ be the solⁿ wrt i.c $\tilde{y}(x_0) = y_0 + \Delta y_0$

we say that solⁿ of $y(x)$ of ① is stable if given $\varepsilon > 0$, $\exists\, \delta > 0$ ($\delta = \delta(\varepsilon, x_0)$) s.t. $|\Delta y_0| < \delta \Rightarrow |y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)| < \varepsilon$.

$\Rightarrow |\tilde{y} - y| < \varepsilon$ → Lyapunov's stability

The solⁿ is asymptotically stable if it is stable and $|y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)| \to 0$ as $x \to \infty$ when $|\Delta y_0| < \delta$.

[stronger condⁿ]

↳ $y' = ay$, $y(x_0) = y_0$

$\Rightarrow y(x) = e^{ax} y_0$

$\tilde{y} = y(x, x_0, y_0 + \Delta y_0) = e^{ax}(y_0 + \Delta y_0)$

$|\tilde{y} - y| = |e^{ax} \Delta y_0| < \varepsilon$ if $a < 0$

Thm: All solⁿs of the diff sys $y' = A(x)y$ are stable $\Leftrightarrow$ they are bdd.

Pf Given that $y' = A(x)y$. → ①

($\Rightarrow$) Let ϕ be FM of ①, WKT any solⁿ $y(x)$ of ① is $y(x) = \phi(x)c$
$= \phi(x)\phi^{-1}(x_0) y_0$.

Suppose that solⁿ is bdd $\Rightarrow \exists$ a $c > 0$ s.t.
$|\phi(x)| < c$

TPT: solⁿ is stable.

→ Consider $|y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)|$

$= |\phi(x)\phi^{-1}(x_0)(y_0 + \Delta y_0) - \phi(x)\phi^{-1}(x_0) y_0|$

$= |\phi(x)\phi^{-1}(x_0) \Delta y_0|$

$\le c|\phi^{-1}(x_0)||\Delta y_0|$

$\le c|\phi^{-1}(x_0)|\delta$

$< \varepsilon$ if $\delta = \dfrac{\varepsilon}{c|\phi^{-1}(x_0)|}$ $\Rightarrow$ The solⁿ is stable

($\Leftarrow$) Conversely, suppose that the solⁿ is stable and we have tpt the solⁿ is bdd.

solⁿ stable $\Rightarrow |y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)| < \varepsilon$ when $|\Delta y_0| < \delta$

$\Rightarrow |\phi(x)\phi^{-1}(x_0)\Delta y_0| < \varepsilon$

$\Rightarrow |\phi(x)||\phi^{-1}(x_0)||\Delta y_0| < \varepsilon$

$\Rightarrow |\phi(x)| < c$ where $c = \varepsilon / \delta |\phi^{-1}(x_0)|$

$\Rightarrow$ solⁿ is bdd//

- **Thm:** Let $\phi(x)$ be the FM of $y' = A(x)\,y$. Then, all solns of diff sys are asymptotically stable $\Leftrightarrow \|\phi(x)\| \to 0$ as $x \to \infty$.

Pf: Given that ϕ is the FM of $y' = A(x)y$

($\Rightarrow$) Suppose $\|\phi(x)\| \to 0$ as $x \to \infty$ and ϕ is cont. Then, $\exists$ const

$$\|\phi(x)\| \le c \Rightarrow \text{soln bdd}.$$

$\Rightarrow$ by prev thm, soln is stable

Consider, $\|y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)\|$

$$= \|\phi(x)\,\phi^{-1}(x_0)\,\Delta y_0\|$$

$$= \|\phi(x)\|\,\|\phi^{-1}(x_0)\Delta y_0\| \to 0 \text{ as } x \to \infty$$

Thus, soln is asymptotically stable.

($\Leftarrow$) Conv, suppose soln is asymptotically stable.

TPT $\|\phi(x)\| \to 0$ as $x \to \infty$

$$AS \Rightarrow \|y(x, x_0, y_0 + \Delta y_0) - y(x, x_0, y_0)\| \to 0 \text{ as } x \to \infty$$

$$\Rightarrow \|\phi(x)\,\phi^{-1}(x_0)\,\Delta y_0\| \to 0 \text{ as } x \to \infty$$

$$\Rightarrow \|\phi(x)\|\,\|\phi^{-1}(x_0)\Delta y_0\| \to 0 \text{ as } x \to \infty$$

$$\Rightarrow \|\phi(x)\| \to 0 \text{ as } x \to \infty.$$

(R 2)

- **Autonomous system:** $y' = f(x, y)$.

$$y' = A(x)y + B(x)$$

If A depends on x, then it is called non-autonomous sys and if A does not depend on x (ie, A is const matrix), then it is known as autonomous sys.

If $B = 0$, then it is called Homogeneous sys, & if $B \neq 0$, then it is non-hom. autonomous sys. $y' = Ay$ A: const matrix.

$$y' = f(x, y)$$

$$y' = f(y), \quad y' = Ay$$

suppose $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$,

we have $y(x) = e^{(x - x_0)A}\,y_0$ | $y' = Ay$, $y(x_0) = y_0$

$$y(x) = \text{diag}\left(e^{\lambda_1(x - x_0)}, e^{\lambda_2(x - x_0)}, \ldots, e^{\lambda_n(x - x_0)}\right) y_0.$$

$$y' = Ay$$

$$y = (y_1, \ldots, y_n)$$

y_i's are known as the phases of the sys.

$\{y(x) \in \mathbb{R}^n\}$ is the trajectory or orbit of sys

$\mathbb{R}^n$: phase space and when $n = 2$, it's called as phase-plane of sys.

$\left[y' = Ay \longrightarrow y' = By \quad B = P^{-1}AP \right]$
red. to diag sys. & solve

26/3/26

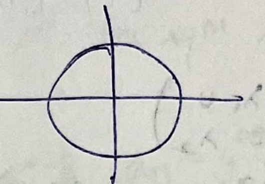
Ex? :

$y' = Ay$

$y_1' = -y_2 \qquad y_2' = y_1$

$y_1'' = -y_2' = -y_1 \Rightarrow y_1'' + y_1 = 0$

$y_1 = \cos x \quad y_2 = \sin x$



Ex?

$y_1' = -y_1 - y_2$

$y_2' = y_1 - y_2$

$y_1'' = -y_1' - y_2'$

$= -y_1' - (y_1 - y_2)$

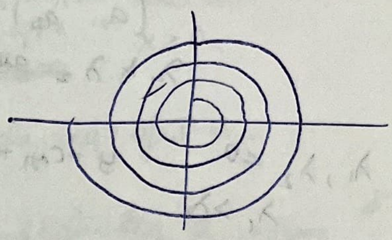
$= -y_1' - y_1 + (-y_1' + y_1)$

$y_1'' + 2y_1' - 2y_1 = 0$

$m = -1 \pm i$

$y_1 = e^{-x}\cos x$

$y_2 = e^{-x}\sin x$



- Eq^n points : $y' = Ay, \quad y' = f(y)$

We say that $\bar{y}$ is an equilibrium pt of $y' = f(y)$ $[y' = Ay]$ if $f(\bar{y}) = 0$.

↳ It is aka Critical pt / singular point $y(x) = \bar{y}$ is a sol^n of $y' = Ay$ if $\bar{y}$ is an equilibrium point.

$y' = Ay, \quad \det(A) \neq 0$

then $y = 0$ is an eq^m pt ; infact, $y = 0$ is the unique eq^m pt.

helps us to analyze stab. of sol^n.

↳ It is aka steady state sol^n //.

- To find the eq^m point :

Solve for $f(\bar{y}) = 0$

(Q.) Find all eq^m points of sys :

(i) $dy_1/dx = 1 - y_2 \qquad dy_2/dx = y_1 + y_2$

$f(y) = \begin{pmatrix} 1 - y_2 \\ y_1 + y_2 \end{pmatrix}$. Let $\bar{y}$ be eq^m pt

$\Rightarrow f(\bar{y}) = 0 \Rightarrow 1 - \bar{y}_2 = 0$

$\bar{y}_1 + \bar{y}_2 = 0$

$\bar{y}_2 = 1 \qquad \bar{y}_1 = -1$

$\therefore \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the eq^m pt

(ii) $y_1' = (y_1-1)(y_2-1)$, $y_2' = (y_1+1)(y_2+1)$

$f(\bar{y}) = 0 \Rightarrow (\bar{y}_1 - 1)(\bar{y}_2 - 1) = 0 \Rightarrow \bar{y}_1 = 1$ or $\bar{y}_2 = 1$

$(\bar{y}_1+1)(\bar{y}_2+1) = 0 \Rightarrow \bar{y}_1 = -1$ or $\bar{y}_2 = -1$

$\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$ are eq^n pts of sys

→ $y' = Ay$, $A = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

$y_1(x) = e^{\lambda_1 x} v_1$ $\quad y_2(x) = e^{\lambda_2 x} v_2$

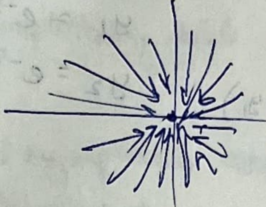
$A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ $\quad |A - \lambda I| = 0$

λ_1 & λ_2 are roots of ch. poly.

→ $\lambda_1, \lambda_2 < 0$ $\quad y = c_1 y_1 + c_2 y_2$

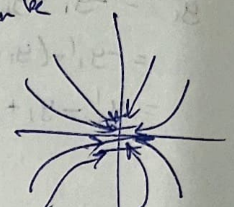
$\lambda_1 > \lambda_2$

$y_1(x) \to 0$
$y_2(x) \to 0$ as $x \to \infty$



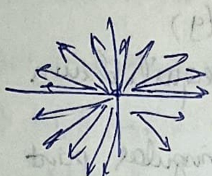
or can be

stable eq^n pt



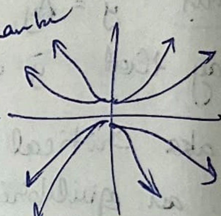
→ when λ_1 & $\lambda_2 > 0$,

$y_1(x) \to \infty$
$y_2(x) \to \infty$ as $x \to \infty$



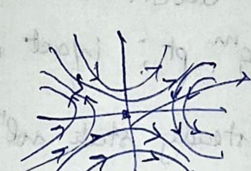
or can be

unstable eq^n pt



→ one +ve, one −ve.

$\frac{y_1}{y_2} = e^{(\lambda_2 - \lambda_1)} \frac{v_1}{v_2}$ $\quad \lambda_2 < 0, \lambda_1 > 0.$



saddle pt
↳ unstable pt

→ Stability of an Autonomous system:

Consider $y' = Ay \to ①$. Let $\phi(x)$ be a sol^n of ①.

→ The sol^n $y = \phi(x)$ is said to be stable if every sol^n $\psi(x)$ of ① starting sufficiently close to $\phi(x)$ at $x=0$ must remain close to $\phi(x)$ in future time as well.

→ If atleast one of the sol^n $\psi(x)$ sufficiently close to $\phi(x)$ at $x=0$ deviates away from $\phi(x)$ as $x \to \infty$, then we say that ϕ is unstable.

→ In other words, stability means given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|\phi_j(x) - \psi_j(x)| < \varepsilon$ whenever $|\phi_j(0) - \psi_j(0)| < \delta$, $j = 1, \dots, n$.

→ Thm: Every sol^n $y = \phi(x)$ of $y' = Ay$ is stable if the eq^n sol^n is stable.

Pf: Let $y(x) = 0$ be the eq^m sol^n of $y' = Ay \to ①$

Let ϕ and ψ be sol^m of ①. Let $z(x) = \phi - \psi$, z is also a sol^n of ①.

Given $y(x) = 0$ is stable

→ $\Sigma(x)$ close to $f(x)$ at $x=0$ remains close to it in future as well

⇒ $\phi(x)$ close to $\psi(x)$ at $x=0$ and remains close to $\psi(x)$ at future time as well

⇒ The solⁿ is stable.

→ Thm: A solⁿ $y=\phi(x)$ of $y'=Ay$ is unstable if the eqᵐ solⁿ is unstable.

Pf: exer.

30/3/26 → Propⁿ: Let all the roots of the char eqⁿ of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$, where a_i's are some constts, have negative real parts and let $f \geq 0$ be s.t $-f > \max_{1 \leq j \leq m} \text{Real}(e^{\lambda_j})$ where $\lambda_1, \dots, \lambda_m$ are distinct roots.

Then, ∃ constt k s.t $|y(x)| \leq k e^{-fx}$, $x \geq 0$.

Pf: We have $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y$. The ch. eqⁿ of ① is:

$\lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$

Suppose $\lambda_1, \dots, \lambda_m$ be the roots with multiplicities $q_1, \dots, q_m$ s.t

$q_1 + \dots + q_m = n$.

Let $\lambda_j = \alpha_j + i\beta_j$ $j = 1, \dots, m$, $\alpha_j < 0$

Let $y_j(x) = x^p e^{\lambda_j x}$ $0 \leq p < q_j$, $j = 1, \dots, m$.

Consider $y_j(x) e^{fx} = x^p e^{\lambda_j x} e^{fx} = x^p e^{(\alpha_j + f + i\beta_j)x}$

Note that $\alpha_j + f < 0$

$|y_j(x) e^{fx}| = |x^p e^{(\alpha_j + f)x}|$, ∃ k_j s.t

$\leq k_j$

⇒ $|y_j(x)| \leq k_j e^{-fx}$

WKT

$y(x) = \Sigma c_i y_i(x)$

$|y(x)| = \Sigma c_i |y_i(x)|$

$\leq \max c_i \Sigma |y_i(x)|$

$\leq \max c_i \Sigma k_i e^{-fx}$

$\leq k e^{-fx}$ (P).

→ Thm: (i) Every solⁿ $y=\phi(x)$ of $y'=Ay$ is stable if all e-vals have negative real part.

(ii) Every solⁿ $y=\phi(x)$ of $y'=Ay$ is unstable if atleast one e-val of A has +ve real part

Pf: We have $y'=Ay$ → ①.

WKT any solⁿ $\psi(x)$ of ① is $\psi(x) = e^{xA}\psi(0)$.

Let ϕ_{ij} be the i-jth component of e^{xA}, then $\psi_i(x) = \sum_{j=1}^{n} \phi_{ij} \psi_j(0)$

$i = 1, \dots, n$.

Pf: (i)
Let us assume that the e-val of A has -ve real part. Let $-\alpha_1$ be the largest -ve real part of such e-vals.

Then, $\exists\, \rho$ s.t. $-\alpha_1 < -\rho < 0$.

By the prev prop^n, $\exists\, k>0$ s.t. $|\phi_{ij}(x)| \le ke^{-\rho x}$.

Define $\|\psi(x)\| = \max\{|\psi_j(x)|, j=1,\dots,n\}$.

$$\begin{aligned}\|\psi(x)\| &= \max|\psi_j(x)| \\ &= \max\left|\textstyle\sum_{j=1}^n \phi_{ij}(x)\psi_j(0)\right| \\ &\le \max\left|\textstyle\sum_{j=1}^n ke^{-\rho x}\psi_j(0)\right| \\ &= nk\|\psi(0)\| \\ &\le nk\,\delta, \quad \|\psi(0)\| < \delta \\ &< \varepsilon \quad \text{if we choose } \delta = \tfrac{\varepsilon}{nk}\end{aligned}$$

$\Rightarrow \|\psi(x)\| < \varepsilon$ when $\|\psi(0)\| < \delta$

$\rightarrow$ Any sol^n $\psi(x)$ sufficiently close to eq^n pt at $x=0$ remains close to it in future as well

$\Rightarrow$ sol^n is stable

(ii)
Let λ be any +ve e-val of A.
The $\psi(x) = ce^{\lambda x}v$, where v is e-vector.

$\|\psi(x)\| = \|ce^{\lambda x}v\| \longrightarrow \infty$ as $x \to \infty$
(regardless of val of c)

$\Rightarrow$ sol^n is unstable

Rd.

(Q) (i) Show that every sol^n of system $y' = Ay$, $A = \begin{pmatrix}1 & 5\\ 5 & 1\end{pmatrix}$ is unstable

(ii) Determine whether each sol^n $y(x)$ of $y' = Ay$, $A = \begin{pmatrix}-1 & 0 & 0\\ -2 & -1 & -2\\ -3 & -2 & -1\end{pmatrix}$ is asymptotically stable or unstable.

Ans (i) $\begin{vmatrix}1-\lambda & 5\\ 5 & 1-\lambda\end{vmatrix} = \lambda^2 - 2\lambda \cancel{+6}\ -24 = 0$.

$\lambda = \dfrac{2 \pm \sqrt{4+96}}{2} = \dfrac{2 \pm 10}{2} = 6, -4$

+ve e-val (re-part)
$\Rightarrow$ sol^n unstable

(ii) $(-1-\lambda)\left((-1-\lambda)^2 - (4)\right) = 0$

$(-1-\lambda)(\lambda^2 + 2\lambda - 3) = 0$

$\lambda = -1, -1 \pm 2i$

$\lambda = \dfrac{-2 \pm \sqrt{4+12}}{2}$

$= \dfrac{-2 \pm 4}{2}$

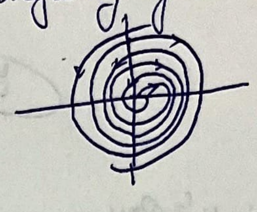
$= -3, 1$

1/4/26

- **Limit Cycle:**

$$\left.\begin{aligned} y_1' &= f_1(x,y) \\ y_2' &= f_2(x,y) \end{aligned}\right\} \quad (1)$$

A closed trajectory of the differential sys. (1) which is approached spirally either from inside or from outside by a non-closed trajectory of (1) as $x \to \infty$ or $x \to -\infty$ is called a lim cycle.



- **Poincare – Bendixson Thm:**

Suppose that a ~~sys~~ solution $y(x) = [y_1(x), y_2(x)]$ of the diff sys $\left.\begin{aligned} y_1' &= f_1(x,y) \\ y_2' &= f_2(x,y) \end{aligned}\right\} (1)$ remains in a bounded region of $y_1 - y_2$ plane which contains no critical/eq^m pts of (1). Then, its trajectory must spiral into a simple closed curve which itself is the trajectory of a periodic sol^n.

Ex: (1) $y'' + (2y^2 + 3y'^2 - 1)y' + y = 0$

Let $\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\}$ $y_1' = y'' = y_2$

$y_2' = y'' = (1 - 2y^2 - 3y'^2)y' - y$

$y_2' = (1 - 2y_1^2 - 3y_2^2)y_2 - y_1$

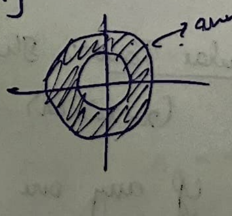
Consider $\dfrac{d(y_1^2 + y_2^2)}{dx} = 2y_1y_1' + 2y_2y_2'$

$= \cancel{2y_1y_2} + 2y_2[(1 - 2y_1^2 - 3y_2^2)y_2 - \cancel{y_1}]$

$= 2y_2^2[1 - 2y_1^2 - 3y_2^2]$

$\gtrless 0 \Rightarrow \frac{1}{3} < y_1^2 + y_2^2 < \frac{1}{2}$

annular.



GATE-CSIR

- **Power series sol^n for d.e:**

$\sum_{0}^{\infty} a_n x^n$, $|x| < R$ for cgs

Ratio test, $\lim \left|\frac{a_{n+1}}{a_n}\right| |x| < 1$

$|x| < \lim \left|\frac{a_n}{a_{n+1}}\right| = R$.

Suppose $y' = y \longrightarrow (1)$

Assume that $y = \sum a_n x^n$ is a sol^n of (1)

$y = a_0 + a_1x + a_2x^2 + \cdots$

$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$

from (1), $a_0 + a_1x + a_2x^2 + \cdots = a_1 + 2a_2x + 3a_3x^2 + \cdots$

$\Rightarrow \quad a_1 = a_0$

$\Rightarrow \quad 2a_2 = a_1 = a_0 \Rightarrow a_2 = a_0/2!$

$\Rightarrow \quad 3a_3 = a_2 = a_0/2 \Rightarrow a_3 = a_0/3!$

$\therefore y = \frac{a_0}{0!} + \frac{a_0}{1!}x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots$

$= a_0\left(1 + x + x^2/2! + \cdots\right)$

$y = a_0 e^x$

Ex: $y' = 2xy$

Assume $y = \sum a_n x^n$ is sol^n.

$2x\left(a_0 + a_1x + a_2x^2 + \cdots\right) = a_1 + 2a_2x + 3a_3x^2 + \cdots$

$2\left[a_0x + a_1x^2 + a_2x^3 + \cdots\right] = a_1 + 2a_2x + 3a_3x^2 + \cdots$

$a_1 = 0$

$2a_0 = 2a_2 = 4a_4x^3$

$2a_1 = 3a_3 = 0$

$2a_2 = 4a_4x^3$

$y = a_0\left(1 + x^2 + \frac{x^4}{2!} + \cdots\right)$

$= a_0 e^{x^2}$

→ Ordinary & singular points:

$y'' + P(x)y' + Q(x)y = 0$

P, Q are const at $x = x_0$. A pt $x = x_0$ is said to be an ordinary pt/regular pt if P & Q are analytic at $x = x_0$

→ Regular Singular Point: The pt $x = x_0$ is said to be a regular singular point if $(x-x_0)P(x)$ & $(x-x_0)^2Q(x)$ are analytic at $x = x_0$.

↳ if any one cond^n is not satisfied, it is irregular singular pt.

(Q)

(i) $(1-x^2)y'' - 2xy' + p(p+1)y = 0$

(ii) $x^2y'' + axy' + by = 0$

(iii) $x^2y'' + xy' + (x^2 - n^2)y = 0$

(iv) $x^3(x-2)y'' + x^3y' + 6y = 0$

Ans (i) $y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$

$P = \frac{-2x}{1-x^2}$, $Q = \frac{p(p+1)}{1-x^2}$; $x = \pm 1$: singular, others : ordinary

Take $x=1$: Consider $(x-1)P(x)$ & $(x-1)^2Q(x)$

$\Rightarrow (x-1)\frac{-2x}{(1-x)(1+x)} = \frac{2x}{1+x} \rightarrow$ analytic at $x=1$

$\Rightarrow (x-1)^2\frac{P(P+1)}{(1-x)^2} = -P(1+P)\left(\frac{1-x}{1+x}\right) \rightarrow$ analytic at $x=1$

$\rightarrow x=1$ is regular singular pt

lll^{ly}

$x=-1$ is a RSP.

(ii) $y^2 + \frac{ax}{x^2}y' + \frac{b}{x^2}y = 0$

$P(x) = a/x \qquad Q = b/x^2$

$x=0$ singular, others : ordinary

Take $xP(x)$ & $x^2Q(x)$.

$\Rightarrow xP(x) = a \rightarrow$ analytic

$\Rightarrow x^2Q(x) = b \rightarrow$ analytic.

$\Rightarrow x=0$ is a RSP

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- Thm: Let x_0 be an ordinary point of the d.e $y'' + p(x)y' + Q(x)y = 0 \rightarrow$ ① & let a_0 & a_1 be arbitrary constts. Then, ∃! ~~a~~ function $y(x)$ that is analytic at x_0 is a solution of ① in a certain nbd of x_0 satisfying the I.C. $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series exp^n of $P(x)$ & $Q(x)$ are valid on an interval $|x-x_0| < R$, then the power series exp^n of the sol^n is also valid in the same interval.

(No pf)

Ex: $y'' + y = 0 \rightarrow$ ①

$P = 0 \quad Q = 1.$

Let $y = \sum_0^\infty a_n x^n$ be a sol^n of ①.

$y' = \sum n a_n x^{n-1}$

$y'' = \sum n(n-1) a_n x^{n-2} = \sum (n+1)(n+2) a_{n+2} x^n$

① $\Rightarrow \sum (n+1)(n+2) a_{n+2} x^n + \sum a_n x^n = 0$

$\Rightarrow \sum \left[(n+1)(n+2)a_{n+2} + a_n\right] x^n = 0$

$\Rightarrow (n+1)(n+2) a_{n+2} + a_n = 0$

$\Rightarrow a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$

$n=0 \quad a_2 = -a_0/1\cdot 2$

$n=1 \quad a_3 = -a_1/2\cdot 3$

$n=2 \quad a_4 = -a_2/3\cdot 4 = \frac{a_0}{4!}$

$n=3 \quad a_5 = \frac{-a_3}{4\cdot 5} = a_1/5!$

we have $y(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

$= a_0 \cos x + a_1 \sin x$

Ex 2: Legendre eq^n: $(1-x^2)y'' - 2xy' + p(p+1)y = 0$, near $x=0$, p: const. → ①

$$P(x) = \frac{-2x}{(1-x^2)} \qquad Q(x) = \frac{p(p+1)}{(1-x^2)}$$

P & Q are analytic in nbd of $x=0$.

∴ we have a valid power series near $x=0$

Let $y = \sum_0^\infty a_n x^n$ be a sol^n of ①

$y' = \sum n a_n x^{n-1}$

$y'' = \sum n(n-1) a_n x^{n-2} \Rightarrow \sum (n+1)(n+2) a_{n+2} x^n$

$x^2 y'' = \sum n(n-1) a_n x^n$

$x y' = \sum n a_n x^n$.

① becomes,

$$\sum (n+1)(n+2) a_{n+2} - \sum n(n-1) a_n x^n - 2\sum n a_n x^n + p(p+1)\sum a_n x^n = 0$$

$\Rightarrow \sum \left[(n+1)(n+2) a_{n+2} - \left(n(n-1) + 2n - p(p+1)\right) a_n\right] x^n = 0$

$\Rightarrow (n+1)(n+2) a_{n+2} - \left((n^2 - p^2) + (n-p)\right) a_n = 0$

$\Rightarrow (n+1)(n+2) a_{n+2} = (n-p)(n+p+1) a_n$

$$a_{n+2} = \frac{-(p-n)(p+n+1) a_n}{(n+1)(n+2)}$$

$\underline{n=0}$ $\quad a_2 = \frac{-p(p+1)}{1 \cdot 2} a_0$

$\underline{n=1}$ $\quad a_3 = \frac{-(p-1)(p+2)}{2 \cdot 3} a_1$

$\underline{n=2}$ $\quad a_4 = -\frac{(p-2)(p+3)}{3 \cdot 4} a_2 = \frac{p(p-2)(p+1)(p+3)}{4!} a_0$

$\underline{n=3}$: $\quad a_5 = \frac{-(p-3)(p+4)}{4 \cdot 5} a_3 = \frac{(p+4)(p-3)(p+2)(p-1)}{5!} a_1$

Here,

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$= a_0\left[1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 + \dots\right] + a_1\left[x - \frac{(p-1)(p+2)}{3!} - \frac{(p-1)(p+2)(p-3)(p+4)}{5!} \dots\right]$

$= a_0 y_1(x) + a_1 (y_2(x))$

$p \to$ even, y_1 finite

$p \to$ odd, y_2 finite.

(Q-3) $y'' + xy' + y = 0$.

(i) find $y_g = a_0 y_1 + a_1 y_2$ as power series

(ii) Show that $y_1(x)$ is series exp^n of $e^{-x^2/2}$ & use this fact to find a second sol^n $y_2(x)$.

$\boxed{\text{Exercise}}$

- Power series exp^n for a RSP :

$x = x_0$ is RSP $\Rightarrow$ $(x-x_0)P(x)$ & $(x-x_0)^2 Q(x)$ are analytic.

Let $x_0 = 0$ for convenience $\Rightarrow$ $xP(x)$ & $x^2 Q(x)$ are analytic at $x=0$.

$\Rightarrow$ $xP(x) = \sum p_n x^n$ & $x^2 Q(x) = \sum q_n x^n$

$y = x^m \sum a_n x^n = \sum a_n x^{n+m}$.

where m is the sol^n of $m(m-1) + mp_0 + q_0 = 0$ is known as the indicial eq^n.

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↳ aka Frobenius Series sol^n Method.

- Thm :- (Frobenius series sol^n for DE with RSP) $\to$ ①

Let $x=0$ be a RSP of de $y'' + P(x)y' + Q(x)y = 0$ and that the power series exp^n $xP(x) = \sum P_n x^n$ & $x^2 Q(x) = \sum q_n x^n$ are valid in the interval $|x| < R$, $R > 0$. Let the indicial eq^n $m(m-1) + mp_0 + q_0 = 0$ have real roots m_1 & m_2 with $m_2 < m_1$. Then, ① has atleast one sol^n $y_1(x) = x^{m_1} \sum a_n x^n$, $a_0 \neq 0$ on the interval $0 < x < R$, where a_n's are determined i.t.o a_0 by the recurrence rel^n

$$a_n [(m+n)(m+n-1) + (m+n)p_0 + q_0] + \sum_{k=0}^{n-1} a_k [(m+k) p_{n-k} + q_{n-k}] = 0 \to ②.$$

with m replaced by m_1 & series $\sum a_n x^n$ cgs for $|x| < R$. Furthermore, if $m_1 - m_2 \neq 0$ or $m_1 - m_2 > 0 \in \mathbb{Z}^+$, then ① has a second independent solution $y_2(x) = x^{m_2} \sum a_n x^n$, $a_0 \neq 0$. On the same interval, where a_n's are determined from ② i.t.o a_0 with m replaced m_2 & series cgs for $|x| < R$.

When $m_1 - m_2 = 0$ or an integer, the second solution can be found by computing $y_2 = v y_1$ where $v' = \frac{1}{y_1^2} e^{-\int P dx}$ and the sol^n will be of form $y_2(x) = y_1 \log x + \sum a_n x^n$

(Q) $2x^2 y'' + x(2x+1)y' - y = 0$: find indicial eq^n & its roots.

$$y'' + \frac{x(2x+1)}{2x^2} y' - \frac{y}{2x^2} = 0$$

$$P(x) = \frac{x(2x+1)}{2x^2} = 1 + \frac{1}{2x} \qquad Q(x) = -\frac{1}{2x^2}.$$

$x=0$ is a singular pt,

$$\left.\begin{aligned} xP(x) &= x + \tfrac{1}{2} \\ x^2 Q(x) &= -\tfrac{1}{2} \end{aligned}\right\} \Rightarrow x = 0 \text{ is a RSP } \& \; p_0 = \tfrac{1}{2} \;\&\; q_0 = -\tfrac{1}{2}$$

$\therefore$ The indicial eq^n is $m(m-1) + mp_0 + q_0 = 0 \Rightarrow m(m-1) + \frac{1}{2}m - \frac{1}{2} = 0$

$\Rightarrow 2m(m-1) + m - 1 = 0$

$\Rightarrow (m-1)(2m+1) = 0$

$\boxed{m = -\frac{1}{2}, 1}$

(Q1) Find two independent Frobenius series sol^n of

(i) $x y'' + 2y' + xy = 0$.

(ii) $x^2y'' - x^2y' + (x^2-2)y = 0$

(Q2) Find a Frobenius series sol^n of $x^2y'' + xy' + x^2y = 0$

$(m_1 = m_2)$

(Q.1.A) (i) $xy'' + 2y' + xy = 0$

$\Rightarrow y'' + \frac{2}{x}y' + y = 0$

$x=0$ is singular, $P = 2/x \quad Q = 1$

$xP(x) = 2, \quad x^2Q(x) = x^2 \Longrightarrow x=0$ is RSP

$p_0 = 2 \quad q_0 = 0.$

indicial eq^n.

$m(m-1) + 2m + 0 = 0 \Rightarrow m(m-1+2) = 0$

$\Rightarrow m = 0, -1$

Let $y = \sum a_n x^{m+n}$ is a sol^n of de $\quad y' = \sum (m+n)a_n x^{m+n-1}$

$y'' = \sum(m+n)(m+n-1)a_n x^{m+n-2}$

$xy'' = \sum(m+n)(m+n-1)a_n x^{m+n-1}$

$xy = \sum a_n x^{m+n+1}$

$\sum\left[(m+n)(m+n+1)a_{n+1} + 2(m+n+1)a_{n+1} + a_{n-1}\right]x^{m+n} = 0$

$(m+n+1)(m+n+2)a_{n+1} = -a_{n-1}$

$a_{n+1} = \dfrac{-a_{n-1}}{(m+n+1)(m+n+2)}$

when $m=0$,

$a_{n+1} = \dfrac{-a_{n-1}}{(n+1)(n+2)} \longrightarrow$ 1st sol^n

$m=-1$

$a_{n+1} = \dfrac{-a_{n-1}}{n(n+1)} \longrightarrow$ 2nd sol^n.

→ **Note** : For equal roots, to get 2 indep solⁿ,

one solⁿ (y_1) will be obt by prev method

for other root,

$$y_2(x) = y \log x + \Sigma a_n x^n$$

$$y_2 = v y_1 \quad , \quad v = \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

$$x P(x) = \Sigma p_n x^n$$

$$p(x) = \frac{p_0}{x} + p_1 + p_2 x + \ldots$$

$$e^{-\int P dx} = e^{-\left[\int \frac{p_0}{x} + p_1 + p_2 x + \ldots\right]}$$

$$= e^{-\left[p_0 \log x + p_1 x + p_2 x^2/2 + \ldots\right]}$$

$$= x^{-p_0} e^{-\left[p_1 x + p_2 x^2/2 + \ldots\right]}$$

$$= x^{-p_0} e^{-[p_1 x + \ldots]}$$

$$v = \int \frac{1}{(x^m \Sigma a_n x^n)^2} = \int \frac{1}{x^{m_1 - p_0} (\Sigma a_n x^n)^2} e^{-[p_1 x + \ldots]}$$

→ **Legendre Eqⁿ** :

$$(1-x^2) y'' - 2xy' + p(p+1) y = 0.$$

$$a_{n+2} = \frac{-(p-n)(p+n+1)}{(n+1)(n+2)} a_n$$

$y = a_0 y_1 + a_1 y_2$ → one ∞ series, other finite deg poly.

↳ The Legendre poly $P_n(x)$ (of deg n) is the solⁿ of the Legendre eqⁿ satisfying

$$P_n(1) = 1.$$

→ **Rodrigues Formula for Legendre Poly** :

The Legendre poly $P_n(x)$ is given by

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$