Jan 5, Lec - 1

→ General theory of IVP
→ linear system
→ non-Linear system
→ BVP
→

weightage:

Endsem	50%
Midsem	30%
Quiz	20%

What about ODE's ?

Real world ⟶ Conceptual world
↓
Observation
↓
Modelling
↓
Prediction

system
↓
observation
↓
model ⟶ test
↓
predict
↓
validate

N denotes the no. of people living in a city at a time t.

$N(t)$ — dep. var (N), indep. var. (t)

Initially, $t = 0$,

$$N(0) = 1000.$$

So at a later time,

$$N(t + \delta t) = N(t) + K(N, t).dt$$

$$K(N,t) = \frac{N(t+\delta t) - N(t)}{\delta t}$$

$$K(N,t) = \lim_{\delta t \to 0} \frac{N(t+\delta t) - N(t)}{\delta t}$$

$$= \frac{dN}{dt}$$

Let $K(N,t) = \sigma N = \frac{dN}{dt}$

$$\therefore N = 1000\, e^{\sigma t}$$

DE
- ODE
  - linear
  - non-linear
- PDE
  - linear
  - nuclear

$y' = y$ (linear)

$y'' + yy' = 2x$ (non linear)

$y'' + 2y' + y = 0$ (linear)

Formation of DE :

$$f(x, y, c) = 0$$

arb. const.

$$y = 2x + c$$

$$\frac{dy}{dx} = 2.$$

$y = mx + c$ (arb. const)

$\frac{dy}{dx} = m$

$\frac{d^2y}{dx^2} = 0.$

Again,

$f(x, y, y') = 0$

implicit and explicit solns.

Jan 7, Lect-2

Previously learnt:

→ Math modelling

→ formulas for DE

→ solns for DE

$F(x, y, y') = 0$

$y = \phi(x)$

$F(x, \phi, \phi') = 0$

$\frac{dy}{dx} = 2x$ $\qquad$ $\frac{d^2y}{dx^2} = 0$

$y = x^2 + c$ $\qquad$ $\frac{dy}{dx} = m$

$y = mx + c$

$F(x, y, y', \ldots, y^{(n)}) = 0$

Soln. contains 'n' arbitrary constants.

↳ General soln. of DE.

→ When assign particular values for arb. const. we get particular soln.

$F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

$F : I \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

any interval

Classification of DE

→ Variable separation

→ Exact equation

→ Homogenous equation

→ Any other reduces to the above.

Separable eqn:

$\frac{dy}{dx} = f(x, y)$

$= g(x) \cdot h(y).$

$\frac{dy}{h(y)} = g(x) \cdot dx$

Soln is obtained by integration.

$F(x, y, y') = 0$

↳ $M(x, y) \cdot dx + N(x, y) \cdot dy = 0$

$f(x) \cdot g(y) \cdot dx + F(x) \cdot G(y)\, dy = 0$

$\frac{f(x)}{F(x)} \cdot dx + \frac{G(y)}{g(y)} \cdot dy = 0$

Integrate to get soln.

Problems:

1. $y' = (x^2 - 4)(3y + 2)$

$\frac{dy}{3y+2} = (x^2 - 4)\, dx$

$\frac{1}{3} \ln(3y+2) = \frac{x^3}{3} - 4x$

$y = \frac{1}{3} e^{x^3 - 12x} - \frac{2}{3}$

2. $(x-4)\, y^4 \cdot dx - x^3 (y^2 - 3)\, dy = 0$

$dx \cdot \frac{(x-4)}{x^3} = dy \cdot \frac{y^2 - 3}{y^4}$

$\frac{-1}{x} + \frac{2}{x^2} = \frac{-1}{y} + \frac{1}{y^3}$

$y^3 (2 - x) = x^3 (1 - y^2)$

Exact DE :

$$F(x, y, y') = 0$$

$$M.dx + N.dy = 0$$

This is said to an exact DE if
$\exists$ a func. $F(x,y)$ s.t.

$$dF = M.dx + N.dy.$$

$$dF = \frac{\partial F}{\partial x}.dx + \frac{\partial F}{\partial y}.dy.$$

So, $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$

Theorem:

Consider the DE,

$$M.dx + N.dy = 0,$$

where M, N are part. derivs at all points (x,y) in a rectangular domain D.

(i) if the DE is exact in D, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \forall (x,y) \in D$$

(ii) conversely, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \forall (x,y) \in D,$$

then the DE is exact.

Proof:

Suppose the DE is exact.

$\Rightarrow$ $\exists$ $F(x,y)$ s.t.

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

$$\frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

So if DE is exact,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Converse:

$\Rightarrow$ Suppose the DE satisfies,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Should show DE is exact.

$\Rightarrow$ T.P.T $\exists$ F s.t. $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$.

$\Rightarrow$ Assume $\exists$ a func, F $\ni$ $\frac{\partial F}{\partial x} = M$.

So,

$$F = \int M.dx + \phi(y).$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M.dx\right] + \frac{\partial \phi}{\partial y}(y)$$

So,

$$N = \frac{\partial F}{\partial y}.$$

$$\frac{\partial \phi}{\partial y}(y) = N - \frac{\partial}{\partial y}\left[\int M.dx\right]$$

$$\phi(y) = \int\left[N - \frac{\partial}{\partial y}\left[\int M.dx\right]\right] dy$$

Since $\phi(y)$ a func. of y,

$\frac{d\phi}{dy}$ is indep. of x.

$$\Rightarrow \quad \frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\left[\int M.dx\right]\right] = 0.$$

$$\Rightarrow \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

There fore,

$$F = \int M.dx + \int\left(N - \frac{\partial}{\partial y}\int M.dx\right) dy$$

$\rightarrow$ The soln. of exact DE
$M\,dx + N\,dy = 0$ is $F(x,y) = C$.

$\rightarrow$ Reduce a non-exact DE into exact by mult. Int. Factor (IF).

Problems:

1. $y^2\,dx + 2xy\,dy = 0$ — Verify if exact
2. $y\,dx + 2x\,dy = 0$
3. $\underbrace{2x.\sin y + y^3 e^x}_{M} . dx + \underbrace{x^2\cos y + 3y^2 e^x}_{N} dy = 0$

Jan 8, Lect - 3.

Solve:

1. $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$
2. $(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$

1. $M\,dx + N\,dy = 0$

$$\frac{\partial M}{\partial y} = 4x, \quad \frac{\partial N}{\partial x} = 4x$$

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

$F_M = x^3 + 2x^2y$

$F_N = 2x^2y + y^2$

$\therefore F = x^3 + 2x^2y + y^2.$

2.
$$\frac{\partial M}{\partial y} = 3 + 8xy$$

$$\frac{\partial N}{\partial x} = 2 + 6xy$$

IF: $e^{\int P(x).dx}$

$$\frac{dy}{dx} = \frac{3y + 4xy^2}{2x + 3x^2y} = \frac{3y + 4xy^2}{x(2 + 3xy)}$$

$$\frac{\partial(M.f(x,y))}{\partial y} = \frac{\partial(N.f(x,y))}{\partial x}$$

$$M.\frac{\partial f}{\partial y} + f.\frac{\partial M}{\partial y} = N.\frac{\partial f}{\partial x} + f.\frac{\partial N}{\partial x}$$

$$(3y + 4xy^2).\frac{\partial f}{\partial y} + f(3 + 8xy) =$$

$$(2x + 3x^2y).\frac{\partial f}{\partial x} + f.(2 + 6xy)$$

.... $f = x^2y$

Homogenous Equations

If $M.dx + N.dy = 0$, is a homog. DE, then change of var. as $y = vx$ transform the DE into separable eq. in the var. $v.dx$.

Proof:

we have $\frac{dy}{dx} = f(x,y)$.

Given that it is a homog eq. and hence we have,

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

$$y = vx.$$

$$\frac{dy}{dx} = v + x.\frac{dv}{dx}.$$

$$\therefore \frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

$$v + x.\frac{dv}{dx} = g(v)$$

$$x.\frac{dv}{dx} = g(v) - v$$

$$\frac{dv}{g(v) - v} = \frac{1}{x}.dx$$

Integrate to get soln.

Problem:

Solve,

$$(x^2 - 3y^2)\,dx + (2xy)\,dy = 0$$

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3v^2 - 1}{2v}$$

$$x.\frac{dv}{dx} = \frac{3v^2 - 1}{2v} - v$$

$$\Rightarrow y^2 - x^2 = cx^3.$$

Initial and Boundary val. problems:

$$F(x, y, y', \ldots\ldots, y^n) = 0.$$

① ———— $y^n = F(x, y, y', \ldots\ldots, y^{n-1})$

$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$
$$\vdots$$
$$y^{n-1}(x_0) = y_{n-1}$$

when $y_0, \ldots, y_{n-1}$ are constants of x_0 is the initial point.

1. Existence
2. Uniqueness
3. Stability

} well-posed problem.

Even if one not satisfied, } ill-posed problem.

Lemma

If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear func., then

$$F(x) = a_1 x_1 + \ldots\ldots + a_n x_n$$

where $a_0, \ldots, a_n$ are constants of

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Proof:

Let $\{e_i\}_{i=1}^n$ be the basis of $\mathbb{R}^n$. Then, any $x \in \mathbb{R}^n$ can be written as,

$$x = x_1 e_1 + x_2 e_2 + \ldots\ldots + x_n e_n$$
$$F(x) = F(e_1 x_1 + \ldots\ldots + x_n e_n)$$
$$= x_1 F(e_1) + \ldots\ldots + x_n F(e_n)$$
$$= x_1 a_1 + \ldots\ldots + x_n a_n.$$

Theorem

If $F(x, y, y', y'', \ldots, y^n) = 0$, $x \in I$ is a homog. lin. eqn of order n defined on I, then it can be written as

$$F(x, y, \ldots y^n) = a_0(x) \cdot y^n + a_1(x) \cdot y^{n-1} + \ldots + a_n y = 0$$

for each $x \in I$ where $a_0, a_1, \ldots, a_n$ are some func. defined on I.

Jan 12, Lect - 4.

Existence and Uniqueness Theorem

Let the following equation,

$$b(x) = a_0(x) y^n + a_1(x) y^{n-1} + \ldots + a_n(x) y^0 \longrightarrow (1)$$

be the n^{th} order linear DE, where $a_1, \ldots, a_n$ and b are cont. real val. functions on a real int., $I = [a, b]$ and $a_0(x) \neq 0 \ \forall x \in I$. Let x_0 be any part of I and $c_0, \ldots, c_{n-1}$ be any arbitrary real const. Then, $\exists$ a unique solution ϕ of (1) such that,

$$\phi(x_0) = c_0, \quad \phi'(x_1) = c_1, \ldots, \phi^{n-1}(x_{n-1}) = c_{n-1}$$

and the solution ϕ is defined over the entire int. I.

Theorem : Superposition Principle

Let $L_n(y) = b_i(x)$, $i = 1, \ldots, k$ be k diff non-homog. linear DE of order n, where

$$L_n(y) = a_0(x) y^n + \ldots + a_n(x) y^0, \quad a_0 \neq 0. \longrightarrow (2)$$

for any $x \in I$, $(a_0, \ldots, a_n)$ are cont. funcs. of x defined on I. Let ϕ_i be a particular soln. of (2). Then, $\sum_i^k c_i \phi_i$ is a particular soln. of $L_n(y) = \sum_i^k c_i b_i(x)$

Proof

Given, ϕ_i is a particular soln. of (2).

$$\Rightarrow L_n(\phi_i) = b_i$$

To prove that $\sum_i c_i \phi_i$ is a soln. of

$$L_n(y) = \sum_i c_i b_i.$$

we have,

$$L_n(y) = a_0 y^n + a_1 y^{n-1} + \dots + a_n y^0$$

Replace y by $\Sigma c_i \phi_i$.

$$\Rightarrow L_n(\Sigma c_i \phi_i) = a_0 (\Sigma c_i \phi_i)^n + \dots + a_n (\Sigma c_i \phi_i)$$
$$= a_0 [c_1 \phi_1^n + c_2 \phi_2^n + \dots + c_k \phi_k^n] + \dots$$

$$\Rightarrow c_1 [a_0 \phi_1^n + a_1 \phi_1^{n-1} + \dots + a_n \phi_1^0] + \dots + c_k [a_0 \phi_k^n + a_1 \phi_k^{n-1} + \dots + a_n \phi_k^0]$$
$$= c_1 b_1 + c_2 b_2 + \dots + c_k b_k$$
$$= \Sigma c_i b_i$$

Hence, $\Sigma c_i \phi_i$ is a soln of $L_n(y) = \Sigma c_i b_i$.

First Order Linear DE

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Case 1: $Q(x) = 0$,

$$y' + py = 0.$$

Theorem:

Consider $y' + py = 0$, where p is a complex constant. If $c \in \mathbb{C}$, then the func. ϕ is defined by $\phi(x) = ce^{-px}$ is a solution, and moreover, every soln. has this form.

Proof:

Let ϕ is is a soln of $y' + py = 0$. —— ①

$$\Leftrightarrow \phi = ce^{-px}.$$

Suppose ϕ is a soln of ①.

$$\Rightarrow \phi' + p\phi = 0.$$
$$\Rightarrow e^{px}(\phi' + p\phi) = 0$$
$$\Rightarrow (e^{px} \cdot \phi)' = 0$$
$$\Rightarrow e^{px} \cdot \phi = c$$
$$\phi = ce^{-px}$$

Conversely, suppose $\phi = ce^{-px}$,

then $\phi' + p\phi = 0$

$$\Rightarrow -pc \cdot e^{-px} + pc \cdot e^{-px}$$
$$\Rightarrow 0$$

$\Rightarrow ce^{-px}$ is a soln of ①.

Case 2: $y' + py = Q(x)$.

Theorem:

Consider the DE, $y' + py = Q(x)$, where $p \in \mathbb{C}$, and Q is a cont. func on the intv. I. If x_0 is a point in I and c is any const., then the func. ϕ defined by

$$\phi(x) = e^{-px} \cdot \int_{x_0}^{x} e^{pt} \cdot Q(t) \cdot dt + c$$

of the given DE and every soln is of this form.

Proof:

Let ϕ be any soln. of $y' + py = Q(x)$.

$$\Rightarrow \phi' + p\phi = Q$$
$$\Rightarrow e^{px}(\phi' + p\phi) = e^{px} \phi(x)$$
$$\Rightarrow (e^{px} \cdot \phi)' = e^{px} \cdot \phi(x)$$

$$\Rightarrow e^{px} \phi = \int e^{pt} \cdot Q(t) \cdot dt + c$$
$$\Rightarrow \phi = e^{-px} \cdot \int e^{px} \cdot Q(t) \cdot dt + ce^{-px}$$

Problems

1. $y' - 2y = 1$
2. $y' + y = e^x$
3. $y' - 2y = x^2 + x$
4. $3y' + y = 2e^{-x}$

Case 3 : $y' + p(x).y = Q(x)$.

Theorem :

If P and Q are cont. funcs. Let A be any cont. func, where $A' = P$. Then the func. ϕ given by

$$\phi(x) = e^{-A(x)} \int_{x_0}^{x} e^{At}. Q(t).dt, \quad x_0 \in I$$

is the soln of the eqn. $y' + py = Q$ on I. $\longrightarrow$ ①

The func. ϕ_i given by,

$$\phi_i(x) = e^{-A(x)}$$ is a soln of the homog. eqn, $y' + p(x).y = 0$.

If c is any constant, then $\phi + c.\phi_1$ is a solution of ① and every soln. of ① has the same form.

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$$\phi(x) = e^{-A(x)} \int_{x_1}^{x} e^{A(t)}. Q(t).dt$$

$$\phi_1 = e^{-Ax}$$

$\phi + c\phi_1$, is the soln. of $y' + P(x).y = Q(x)$.

Proof :

$$y' + P(x).y = Q(x) \qquad ①$$

let ϕ be the solution.

We are interested in finding a 'u' such that $(u\phi)'$ is integrable,

$$(u\phi)' = u(\phi' + p\phi)$$

Given that $A(x)$ is a function, where $A' = P$,

$$\Rightarrow A = \int P(x).dx$$

let $u = e^A$.

$$(u\phi)' = (e^A\phi)'$$

$$\Rightarrow (e^A\phi)' = e^A\phi' + \phi e^A . A'$$
$$= e^A(\phi' + p\phi)$$
$$= e^A Q(x)$$

$$\Rightarrow (e^A\phi)' = e^A Q(x)$$
$$\therefore e^A\phi = \int e^A . Q(x).dx + C$$

$$\Rightarrow \phi = e^{-A}\int e^A . Q.dx + ce^{-A}$$

If $Q = 0$, then $\phi = ce^{-A} = \phi_1$, is the soln of $y' + py = 0$.

If $c = 0$, then $\phi = e^{-A}\int e^{-A}.Q.dx$ is a partc. soln of ①

$\therefore$ $y = \phi + \phi_1$ is the gen. soln.

$\Rightarrow$ GS = soln of homog. eq + partcl. soln.

Problems

Find all the solutions.

1. $y' + 2xy = x$
2. $y' + \cos(x).y = \sin x . \cos x$
3. $y' + e^x y = 3e^x$

1. $y' + 2xy = x$.

$$P = 2x, \qquad Q = x$$

$$A = \int P.dx = x^2 = \text{IF}.$$

$$GS = \phi = e^{-A}\int e^A . Q(x).dx$$
$$= e^{-x^2}\int e^{x^2}.x \, dx + ce^{-A}$$
$$= e^{-x^2} . \frac{e^{x^2}}{2} + ce^{-A}$$

$$= \frac{1}{2} + ce^{-x^2}$$

2. $y' + \cos(x).y = \sin x.\cos x$

$$P = \cos x, \quad Q = \frac{\sin 2x}{2}$$

$$A = \int P.dx = \sin x = IF$$

GS ⇒ $$\phi = e^{-A}\int e^{A}.Q\,dx + ce^{-A}$$
$$= e^{-\sin x}\int e^{\sin x}.\sin x\cos x.\,dx + ce^{-\sin x}$$

$$\sin x = a$$
$$da = \cos x.\,dx$$
$$e^{a}.a.da = x e^{x}\,dx$$
$$= e^{x}(x-1)$$

$$= e^{-\sin x}\left[e^{\sin x}(\sin x - 1)\right] + ce^{-\sin x}$$
$$= (\sin x - 1) + ce^{-\sin x}$$

3. $y' + e^{x}.y = 3e^{x}$

$$P = e^{x}, \quad Q = 3e^{x}$$
$$A = e^{x}$$

GS = $$\phi = e^{-A}\int e^{-A}.Q(x).dx + ce^{-A}$$
$$= e^{-e^{x}}\int e^{+e^{x}}.3e^{x}\,dx + ce^{-e^{x}}$$
$$= e^{-e^{x}}.3e^{e^{x}} + ce^{-e^{x}}$$
$$= 3 + ce^{-e^{x}}$$

Consider the DE,
$$y' + \cos x.y = e^{-\sin x}$$

1. Find the soln ϕ, which satisfies $\phi(\pi) = \pi$.
2. Show that any soln. ϕ has the property that $\phi(\pi k) - \phi(0) = \pi k$.

Soln:

Comparing the given DE with
$$y' + P(x).y = Q(x),$$

$$P = \cos x, \quad A = \sin x$$
$$Q = e^{-\sin x}.$$

$$\phi = e^{-\sin x}\int e^{\sin x}.e^{-\sin x}.dx + ce^{-\sin x}$$
$$= e^{-\sin x}(x + c)$$

(i) $\phi(\pi) = \pi + c$

$\therefore \pi = \pi + c \Rightarrow c = 0.$

$$\therefore \phi(x) = xe^{-\sin x}$$

(ii) $\phi(\pi k) = \pi k + c$

$\phi(0) = c$

$\therefore \phi(\pi k) - \phi(0) = \pi k$

Bernoulli's equation

⇒ $$y' + P(x).y = Q(x).y^{n}$$

Theorem :

Suppose $n \neq 0$, $n \neq 1$, then the transformation, $v = y^{1-n}$ reduces the Bern. Eq to a linear eqn in v.

Proof

We have,
$$y' + P(x).y = Q(x).y^{n} \quad \text{— (1)}$$

⇒ $$y^{-n}.y' + P(x).y^{1-n} = Q(x) \quad \text{— (2)}$$

Let $v = y^{1-n}$.

$$\frac{dv}{dx} = (1-n).y^{-n}.\frac{dy}{dx}$$

$$\frac{1}{1-n}.\frac{dv}{dx} = y^{-n}.\frac{dy}{dx}$$

② → $\frac{1}{1-n} \cdot \frac{dv}{dx} + Pv = Q$

$$\frac{dv}{dx} + \underbrace{(1-n)P}_{}.v = \underbrace{(1-n).Q}_{}$$

$$\Rightarrow \frac{dv}{dx} + P_1 v = Q_1$$

Problem

Solve, $\frac{dy}{dx} + y = xy^3$

Soln:

$$v = y^{1-3} = y^{-2}$$

$$\therefore \frac{-1}{3-1} \cdot \frac{dv}{dx} = y^{-n} \cdot \frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx}.$$

$$\therefore \frac{dv}{dx} - 2v = 2x$$

Riccati Equation

$$y' = P(x).y^2 + q(x).y + r(x)$$

$y = y_1 + z^{-1}$ will transform the eqn into 2.

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$$y' = p(x).y^2 + q(x).y + r(x) \quad \text{—— ①}$$

$y = y_1 + z^{-1}$ will reduce the eqn to a linear eqn in 2.

$$\Rightarrow y = y_1 + z^{-1}$$

$$y' = y_1' - \frac{z'}{z^2}$$

Then, ① → $y_1' - \frac{z'}{z^2} = P(y_1 + z^{-1})^2 + q(y_1 + z^{-1}) + r$

$$= (py_1^2 + qy_1 + r) + 2py_1z^{-1} + pz^{-2} + qz^{-1}$$

$$= y_1' + z^{-2}(2py_1z + p + \frac{q}{2})$$

$$\Rightarrow -z' = qz + 2py_1z + p$$

$$= z(2py_1 + q) + p$$

$$\Rightarrow \frac{dz}{dx} + (2py_1 + q)z = -p \Rightarrow \frac{dz}{dx} + Pz = Q.$$

Problem.

$$y' = y^2 - 2y + 1$$

$p = 1, \quad q = -2, \quad r = 1.$

$y = 1$ is par. f. soln of ①.

$\therefore y = y_1 + z^{-1}$ will transform eqn ① to linear,

$$\frac{dz}{dx} + Pz = Q, \quad P = 2Py_1 + q$$
$$Q = -p.$$

$$P = 2py_1 + q = 2 - 2 = 0$$
$$Q = -1$$

$$\frac{dz}{dx} = -1, \Rightarrow z = -x + C = c - x$$

$\therefore$ The soln is $y = y_1 + z^{-1}$

$$= 1 + \frac{1}{c-x}$$

Theorem :

Let $\phi_1, \phi_2, \ldots, \phi_n$ be n- functions defined on I, then we say that,

$\phi_1, \ldots, \phi_n$ are LD of $\exists$ const. c_i, not all zeroes such that,

$$c_1.\phi_1 + c_2.\phi_2 + \ldots + c_n\phi_n = 0.$$

Let $\phi_1, \phi_2, \ldots, \phi_n$ be n functions defined on I, and suppose that they are diffr. (n-1) times, then the n^{th} order determinant

$$\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & & \\ \phi_1^{n-1} & \phi_2^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix}$$

is known as Wronskian of $\phi_1, \ldots, \phi_n$ and given by $W(\phi_1, \ldots, \phi_n)(x)$

Theorem:

Let $\phi_1, \phi_2, \dots, \phi_n$ be a set of n funcs defined on an interval $I \subseteq \mathbb{R}$, and let each func. be diffr. $n-1$ times on I.

If the set of n functions are LD on I, then the Wronskian,

$$W(\phi_1, \dots, \phi_n)(x) = 0 \quad \text{on } I.$$

Proof

Given: $\phi_1, \phi_2, \dots\dots, \phi_n$ are LD.

so,

$\Rightarrow \exists\, c_i$ (not all zeros), s.t.

$$c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n = 0.$$

Since ϕ_i are diffrntiable, we have

$$c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n = 0$$
$$c_1\phi_1' + c_2\phi_2' + \dots + c_n\phi_n' = 0$$
$$\vdots$$
$$c_1\phi_1^{n-1} + c_2\phi_2^{n-1} + \dots + c_n\phi_n^{n-1} = 0.$$

For any $x = x_0$, so $\phi_1(x_0), \phi_2(x_0), \dots, \phi_n(x_0)$

We know that c_i are not all zero.

(A) is a syst. of eqn having non-zero soln c_i.

This is possible, if

$$\Rightarrow \begin{vmatrix} \phi_1(x_0) & \dots\dots & \phi_n(x) \\ \vdots & & \\ \phi_1^{n-1}(x_0) & \dots\dots & \phi_n^{n-1}(x_0) \end{vmatrix} = 0.$$

$$\Rightarrow W(\phi_1, \phi_2, \dots, \phi_n) = 0.$$

Since x_0 is arbitrary, we have,

$$W(\phi_1, \phi_2, \dots, \phi_n)(x) = 0, \quad x \in I.$$

Problem

$$\phi_1 = x^2, \quad \phi_2 = x|x|$$

$W(\phi_1, \phi_2) = 0$, but $\phi_1, \phi_2 \subset I$.

Suppose ϕ_1, ϕ_2 s.t.

$\phi_2(x) \neq 0$ on I,

$W(\phi_1, \phi_2) = 0$.

$$d\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\phi_1' - \phi_1\phi_2'}{\phi_2^2} = \frac{W(\phi_1, \phi_2)}{\phi_2^2}$$
$$= 0.$$

$\therefore \phi_1 = c\phi_2$

$\Rightarrow \phi_1$ & ϕ_2 are LD.

Second order linear Homog. eqn:

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

Theorem:

Let a_1, a_2 be constants and consider the eqn,

$$L(y) = y'' + a_1 y' + a_2 y = 0.$$

If r_1, r_2 are distinct. roots of the charac. polyn.,

$P(r) = r^2 + a_1 r + a_2$, then the func., ϕ_1, ϕ_2 defined by,

$$\phi_1(x) = e^{r_1 x},$$
$$\phi_2(x) = e^{r_2 x},$$

are soln of $L(y) = 0$.

If r_1 is repeated soln of $p(r)$, then the func., ϕ_1, ϕ_2,

$$\phi_1(x) = e^{r_1 x}, \quad \phi_2(x) = x e^{r_1 x}$$

are soln. of $L(y) = 0$.

Proof :

Suppose, we have a first order DE,

$\Rightarrow$ $y' + ay = 0$, then we know that, ce^{-ax} is a soln. where $(-a)$ is the soln. of $r + a = 0$.

We know that if we differ. e^{rx} any times, we get only a const. mult. of e^{rx}. This gives an indication that e^{rx} could be a soln. of $L(y) = 0$ for some r.

Consider,

$$L(e^{rx}) = (r^2 + a_1 r + a_2) e^{rx}$$
$$= P(r) . e^{rx}$$

e^{rx} could be a soln of $y'' + a_1 y + a_2 y = 0$.

$\Leftrightarrow$ $P(r) = 0$.

By fundm. theorem, $P(r)$ should have two roots, say r_1, r_2.

Case 1: Suppose $r_1 \neq r_2$.

Then $e^{r_1 x}$ & $e^{r_2 x}$ are soln of $L(y) = 0$.

Case 2: Suppose $r_1 = r_2 = r$.

$\Rightarrow$ $p(r) = 0$, $p'(r) = 0$.

Consider, $\frac{\partial}{\partial r} L(e^{rx}) = L . \left[\frac{\partial}{\partial r}(e^{rx})\right]$

$$= L(xe^{rx})$$

$$L(e^{rx}) \Rightarrow P(r). e^{rx}$$

$$\Rightarrow P'(r) e^{rx} + p(r). xe^{rx} = L(xe^{rx})$$
$$e^{rx}(P'(r) + P(r). x) = L(xe^{rx}).$$

Since $P'(r) = P(r) = 0$,

$$L(xe^{rx}) = 0.$$

$\therefore$ xe^{rx} is soln of eqn.

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Wronskein :

$$W(\phi_1, \phi_2, \ldots\ldots, \phi_n) = \begin{vmatrix} \phi_1 & \ldots & \ldots & \phi_n \\ \phi_1' & \ldots & \ldots & \phi_n' \\ \vdots & & & \\ \phi_1^{n-1} & \ldots & \ldots & \phi_n^{n-1} \end{vmatrix}$$

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

$\phi_1 = e^{r_1 x}$	$\phi_1 = e^{rx}$
$\phi_2 = e^{r_2 x}$	$\phi_2 = xe^{rx}$

Linear independency :

Case 1 : $r_1 \neq r_2$, $\phi_1 = e^{r_1 x}$, $\phi_2 = e^{r_2 x}$.

Suppose,
$$c_1 \phi_1 + c_2 \phi_2 = 0$$
$$c e^{r_1 x} + c_2 e^{r_2 x} = 0$$
$$c_1 + c_2 e^{(r_2 - r_1)x} = 0$$
$$\frac{d}{dx} : \quad c_2 (r_2 - r_1) e^{(r_2 - r_1)x} = 0$$
$$\therefore \quad c_2 = 0$$

$\therefore$ ϕ_1 & ϕ_2 are linearly indep.

Case 2 : $\phi_1 = e^{rx}$, $\phi_2 = xe^{rx}$

Consider,
$$c_1 \phi_1 + c_2 \phi_2 = 0$$
$$c_1 e^{rx} + c_2 xe^{rx} = 0$$
$$c_1 + c_2 x = 0$$
$$\frac{d}{dx} : \quad c_2 = 0.$$
$$\therefore \quad c_1 = 0.$$

$\Rightarrow$ ϕ_1 & ϕ_2 are LI.

Remark : If ϕ_1 & ϕ_2 are solns of $L(y) = 0$, then $c_1\phi_1 + c_2\phi_2$ is also a soln of $L(y) = 0$.

ϕ_1 & ϕ_2 are soln $\Rightarrow$ $L(\phi_1) = L(\phi_2) = 0$.

Consider $L(c_1\phi_1 + c_2\phi_2) = c_1 L(\phi_1) + c_2 L(\phi_2) = 0$

$\therefore$ $c_1\phi_1 + c_2\phi_2$ is a soln.

If r_1 & r_2 are real, then

$$e^{r_1 x} \text{ \& } e^{r_2 x} \text{ are soln.}$$

Suppose r is complex, then we have

$$r_1 = a + ib$$
$$r_2 = a - ib$$

$$e^{r_1 x} = e^{(a+ib)x} = e^{ax}(\cos bx + i\sin bx)$$
$$e^{r_2 x} = e^{ax}(\cos bx - i\sin bx)$$

$$\frac{e^{r_1 x} + e^{r_2 x}}{2} = e^{ax}.\cos bx$$

$$\frac{e^{r_1 x} - e^{r_2 x}}{2} = e^{ax}.\sin bx$$

Norm of a Soln:

Let ϕ be a soln of $L(y) = 0$.
The norm of the soln ϕ is defined by, $\|\phi(x)\|$.

The size of the answer radius ? is defined by,

$$K = 1 + |a_1| + |a_2|.$$

Theorem :

Let ϕ be any soln of

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

on an interval I containing a point x_0, then for all $x \in I$.

$$\|\phi(x_0)\| e^{-K|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{K|x-x_0|}$$

where, $\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{1/2}$ and

$$K = 1 + |a_1| + |a_2|$$

Proof :

Let
$$u(x) = \|\phi(x)\|^2$$
$$= |\phi(x)|^2 + |\phi'(x)|^2$$
$$= \phi\bar{\phi} + \phi'\bar{\phi}'$$

$$\Rightarrow u'(x) = \phi\bar{\phi}' + \phi'\bar{\phi} + \phi'\bar{\phi}'' + \phi''\bar{\phi}'$$
$$|u'(x)| \le 2|\phi'||\phi| + 2|\phi'||\phi''|$$

We know that ϕ is a soln of $L(y) = 0$.

$$\Rightarrow L(\phi) = 0$$
$$\Rightarrow \phi'' + a_1\phi' + a_2\phi = 0$$
$$\phi'' = -(a_1\phi' + a_2\phi)$$
$$\Rightarrow |\phi''| \le |a_1||\phi'| + |a_2||\phi|$$

Therefore,

$$|u'(x)| \le 2|\phi||\phi'| + 2|\phi'|\left[|a_1||\phi'| + |a_2||\phi|\right]$$
$$= 2(1 + |a_2|)|\phi||\phi'| + 2|a_1||\phi'|^2$$
$$\le (1 + |a_2|)(|\phi|^2 + |\phi'|^2) + \ldots$$
$$\le 2(1 + |a_1| + |a_2|)(|\phi|^2 + |\phi'|^2)$$
$$|u'(x)| \le 2.K.u(x).$$

$$\Rightarrow -2K.u(x) \le u'(x) \le 2K.u(x) \quad \text{(A)}$$
$$u'(x) \le 2K.u(x)$$

$$\Rightarrow e^{-2Kx}(u' - 2K.u(x)) \le 0$$
$$(e^{-2Kx}.u)' \le 0$$

For $x_0 < x$, integrate from x_0 to x.

$$e^{-2Kx}.u(x) - e^{-2Kx_0}.u(x_0) \le 0$$
$$\Rightarrow e^{-2Kx}.u(x) \le e^{-2Kx_0}.u(x_0)$$
$$u(x) \le e^{2K(x-x_0)}.u(x_0)$$

$$\Rightarrow \|\phi(x)\|^2 \le \|\phi(x_0)\|^2 e^{2K(x-x_0)}$$

$$\Rightarrow \|\phi(x)\| \le \|\phi(x_0)\| e^{K(x-x_0)}$$

By considering LHS of (A), we get

$$e^{-K(x-x_0)}.\|\phi(x_0)\| \le \|\phi(x)\|$$

Then, we have

$$\|\phi(x_0)\| e^{-k(x-x_0)} \le \|\phi(x)\| \le \|\phi(x_0)\| . e^{k(x-x_0)} \longrightarrow (1)$$

Suppose $x_0 > x$, then

$$\|\phi(x_0)\| e^{k(x-x_0)} \le \|\phi(x)\| \le \|\phi(x_0)\| . e^{-k(x-x_0)} \longrightarrow (2)$$

From (1) & (2), we get the requested result.

Existence theorem for IVP

For any real number, x_0 and constants α, β, there exists a solution ϕ for the IVP,

$$L(y) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta$$
$$-\infty < x < \beta$$

Proof:

Let ϕ_1 & ϕ_2 be the soln of $L(y) = 0$.

We claim that there exists c_1, c_2 such that,

$c_1\phi_1 + c_2\phi_2$ be a soln of the IVP,

$$L(y) = 0$$
$$y(x_0) = \alpha$$
$$y'(x_0) = \beta$$

$\Rightarrow$ $c_1\phi_1 + c_2\phi_2$ is a soln of the IVP.

$\Leftrightarrow$ $c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$
$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta$

It has a unique soln,

$$\Leftrightarrow \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} \ne 0$$

Case 1:

$$\phi_1 = e^{r_1 x}, \quad \phi_2 = e^{r_2 x}, \quad r_1 \ne r_2$$
$$\phi_1' = r_1 e^{r_1 x}, \quad \phi_2' = r_2 e^{r_2 x}$$

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \begin{vmatrix} e^{r_1 x_0} & e^{r_2 x_0} \\ r_1 e^{r_1 x_0} & r_2 e^{r_2 x_0} \end{vmatrix}$$
$$= r_2 . e^{r_1 x_0} . e^{r_2 x_0} - r_1 e^{r_1 x_0} . e^{r_2 x_0}$$
$$= (r_2 - r_1) e^{x_0(r_1 + r_2)} \ne 0$$

Case 2:

$$\phi_1 = e^{rx}, \quad \phi_2 = xe^{rx}$$
$$\phi_1' = re^{rx}, \quad \phi_2' = e^{rx} + rxe^{rx}$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx} + rxe^{rx} \end{vmatrix}$$
$$= e^{rx}(e^{rx} + rxe^{rx}) - (x.r) e^{rx} . e^{rx}$$
$$= e^{2rx}(1 + rx - rx)$$
$$= e^{2rx}$$

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Uniqueness theorem for IVP

Let α, β be any two constants and $x_0 \in \mathbb{R}$ on any interval I containing x_0. Then $\exists$ atmost one soln ϕ of the IVP, $L(y) = 0$, $y(x_0) = \alpha$, $y'(x_0) = \beta$.

Proof:

let ϕ and ψ be the two solns of

$$L(y) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta. \quad \text{———} (1)$$

ϕ is a soln of (1).

$\Rightarrow$ $L(\phi) = 0$, $\phi(x_0) = \alpha$,
$\phi'(x_0) = \beta$.

ψ is a soln of (1),

$\Rightarrow$ $L(\psi) = 0$, $\psi(x_0) = \alpha$,
$\psi'(x_0) = \beta$.

Let $\mu = \phi - \psi$.

$\therefore L(\mu) = L(\phi - \psi)$

$= L(\phi) - L(\psi) = 0$

$\mu(x_0) = (\phi - \psi)(x_0) = 0$

$\mu'(x_0) = 0$

$\|\mu(x)\| \leq \|\mu(x_0)\| \cdot e^{\kappa|x - x_0|}$

$\Rightarrow \|\mu(x)\| = 0$

$\Rightarrow \phi = \psi$

Problems :

1. solve,

$y'' - 2y' - 3y = 0$

$y(0) = 0$

$y'(0) = 1$

2. Consider the eqn.

$y'' + y' - 6y = 0.$

a) compute the soln ϕ satisfying

$\phi(0) = 1, \ \phi'(0) = 0$

b) Compute the soln ψ satisfying

$\psi(0) = 0, \ \psi'(0) = 1$

c) Compute $\psi(1)$ & $\phi(1)$.

Solutions :

1. $y'' - 2y' - 3y = 0, \ y(0) = 0, \ y'(0) = 1.$

Char. eq: $r^2 - 2r - 3 = 0$

$r = 3, -1$

Soln: $y(x) = c_1 e^{-x} + c_2 e^{3x}$

$y(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

$y'(0) = -c_1 + 3c_2 = 1 \Rightarrow c_2 = 1/4$

$c_1 = -1/4$

2. Char. eq:

$r^2 + r - 6 = 0$

$r = -3, +2$

Soln $\Rightarrow y(x) = c_1 e^{-3x} + c_2 e^{2x}$

$\phi(0) = c_1 + c_2 = 1, \quad c_1 = 1 - c_2$

$\phi'(0) = -3c_1 + 2c_2 = 0,$

$-3 + 3c_2 + 2c_2 = 0$

$c_2 = 3/5, \quad c_1 = 2/5$

$\phi(x) = \frac{2}{5} e^{-3x} + \frac{3}{5} e^{2x}$

Similarly,

$\psi(x) = \frac{-1}{5} e^{-3x} + \frac{1}{5} e^{2x}.$

n^{th} order homog. linear eqn with constant coeff.

Theorem :

Consider the n^{th} order linear eqn with const. coeff,

$L(y) = a_0 y^{n} + a_1 y^{n-1} + \ldots + a_n y = 0$

(i) if $r_1, r_2, \ldots, r_n$ are n distinct roots of the char. poly $P(r)$, then

$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ are

LI solns of $L(y) = 0$.

(ii) If $r_1, r_2, \ldots, r_n$ be distinct roots of the char. polyn. $P(r)$, and each r_i is of multiplicity m_i, with

$m_1 + m_2 + \ldots + m_s = n.$

Then,

$\Rightarrow e^{r_1 x}, xe^{r_1 x}, \ldots, x^{m-1} e^{r_1 x}$

$\vdots$

$\Rightarrow e^{r_s x}, xe^{r_s x}, \ldots, x^{m-1} e^{r_s x}$

are LI solns of $L(y)=0$.

Proof:

Consider $L(e^{rx}) = p(r)\cdot e^{rx}$

When, $p(r) = a_0 r^n + a_1 r^{n-1} + \dots + a_n = 0$

(i) If $P(r)$ has n distinct roots $r_1, r_2, \dots, r_n$, then $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$ are the soln of $L(y)=0$.

(ii) Suppose r_i is a root of $P(r)$ with multiplicity m_i,

$$P(r_i) = p'(r_i) = p''(r_i) = \dots = p^{m_i-1}(r_i) = 0$$

Claim: $x^k e^{r_i x}$ is a soln of $L(y)=0$, $k = 1, \dots, m_i - 1$.

Consider, $\dfrac{\partial^k}{\partial r^k}\left(L(e^{rx})\right) = L(r^k e^{rx})\Big|_{r=r_i}$

But $\dfrac{\partial^k}{\partial r^k}\left(L(e^{r_i x})\right) = \dfrac{\partial^k}{\partial r^k}\left(p(r)\cdot e^{rx}\right)$

$$= \left[p^k(r) + {}^kC_1 p^{k-1}(r)x + {}^kC_2 p^{k-2}(r)x^2 + \dots + p(r)x^k\right]e^{rx}$$

$(k = 1, \dots, m-1)$

$= 0$

$\Rightarrow L(x^k e^{rx}) = 0 \Big|_{r=r_i}$

$\Rightarrow x^k e^{rx}$ is a soln of $L(y)=0$.

This is true for any r_i.

Theorem

Let ϕ be the soln of

$$L(y) = y^n + a_1 y^{n-1} + \dots + a_n y = 0$$

on an interval I containing a point $x_0 \in I$. Then for all $x \in I$,

$$\|\phi(x)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x)\| e^{k|x-x_0|}$$

where, $k = 1 + |a_1| + |a_2| + \dots + |a_n|$

Proof:

Let $u(x) = \|\phi(x)\|^2$

$$= |\phi(x)|^2 + |\phi'(x)|^2 + \dots + |\phi^{n-1}(x)|^2$$

$$= \phi\bar{\phi} + \phi'\bar{\phi}' + \dots + \phi^{n-1}\overline{\phi^{n-1}}$$

$$u'(x) = \phi\bar{\phi}' + \phi'\bar{\phi} + \dots + \phi^n\overline{\phi^{n-1}} + \phi^{n-1}\overline{\phi^n}$$

$$|u'(x)| \le 2|\phi||\phi'| + \dots + 2|\phi^n||\phi^{n-1}| \quad \text{——— (A)}$$

Since ϕ is a soln of $L(y)=0$, we have

$$\phi^n + a_1\phi^{n-1} + \dots + a_n\phi = 0$$

$\Rightarrow \phi^n = -(a_1\phi^{n-1} + \dots + a_n\phi)$

$\Rightarrow |\phi^n| \le |a_1||\phi^{n-1}| + \dots + |a_n||\phi|$

Subs. $|\phi^n|$ in (A).

$$|u'(x)| \le 2|\phi||\phi'| + \dots + 2|\phi^{n-1}|\left[|a_1||\phi^{n-1}| + \dots + |a_n||\phi|\right]$$

$$\le 2(1 + |a_1| + \dots + |a_n|)\left[|\phi|^2 + |\phi'|^2 + \dots + |\phi^{n-1}|^2\right]$$

$$= 2k\cdot u(x)$$

$\Rightarrow -2k\cdot u(x) \le u'(x) \le 2k\cdot u(x)$.

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Existence theorem

Let α_1, α_2 be n constants and let x_0 be any real number. Then there exists a soln. ϕ of $L(y)=0$ on $-\infty < x < \infty$ satisfying,

$$\phi(x_0) = \alpha_1,$$
$$\phi'(x_0) = \alpha_2, \dots$$
$$\dots, \phi^{n-1}(x_0) = \alpha_n.$$

Proof :

Let $\phi_1, \phi_2, \dots, \phi_n$ be any n LI soln of $L(y) = 0$. We will show that $\exists$ n unique constants $c_1, c_2, \dots, c_n$ such that $\phi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$ is a soln of $L(y) = 0$, satisfying $\phi(x_0) = \alpha_1$, $\phi'(x_0) = \alpha_2$, $\dots$ $\phi^{n-1}(x_0) = \alpha_n$.

So,
$$\phi(x_0) = \alpha_1$$
$$\vdots$$
$$\phi^{n-1}(x_0) = \alpha_n$$

$$\Rightarrow \left.\begin{aligned} c_1\phi_1(x_0) + \dots + c_n\phi_n(x_0) &= \alpha_1 \\ c_1\phi_1'(x_0) + \dots + c_n\phi_n'(x_0) &= \alpha_2 \\ &\vdots \\ c_1\phi_1^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0) &= \alpha_n \end{aligned}\right\} (A)$$

The determinant of A is $W(\phi_1, \phi_2, \dots, \phi_n)$ and $W \neq 0$, since $\phi_1, \dots, \phi_n$ are LI.

$\Rightarrow$ $\exists$ unique soln $c_1, c_2, \dots, c_n$ such that,

$$\phi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$$

is the soln of the IVP.

Uniqueness theorem

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n constants and let x_0 be any real number on any I interval containing x_0. Then $\exists$ atmost one soln ϕ of $L(y) = 0$ satisfying

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \dots, \quad \phi^{n-1}(x_0) = \alpha_n.$$

Proof

Let ϕ and ψ be any two solutions of IVP $L(y) = 0$, $y(x_0) = \alpha_1, \dots, y^{n-1}(x_0) = \alpha_n$.

Let $\eta = \phi - \psi$.

So,

$$L(\eta) = L(\phi - \psi) = L(\phi) - L(\psi)$$
$$= 0$$

$$\eta(x_0) = \phi(x_0) - \psi(x_0) = 0$$

and,

$$\|\eta(x)\| \le \|\eta(x_0)\| e^{k|x - x_0|}$$

where $k = 1 + |a_1| + \dots + |a_n|$

$$\Rightarrow \|\eta(x)\| \le 0$$
$$\Rightarrow \eta(x) = 0$$
$$\therefore \phi = \psi$$

Equations with variable coeffs

Consider,

$$y'' + P(x).y' + Q(x).y = R(x)$$
$$y'' + P(x).y + Q(x).y =: L(y)$$

then $R(x) = L(y)$.

Theorem :

If y_n is the general soln of
$$y'' + P(x).y' + Q(x).y = 0$$
and y_p is a particular soln of
$$y'' + P(x)y' + Q(x).y = R(x),$$
then

$y_p + y_g$ is the gen. soln of
$$y'' + P(x)y' + Q(x).y = R(x).$$

Proof :

Given that y_n is a soln of $L(y) = 0$,

$$\Rightarrow L(y_n) = 0.$$

y_p is a soln of $L(y) = R(x)$

$$\Rightarrow L(y_p) = R(x)$$

Let $y_g = y_n + y_p$

Consider,

$$L(y_g) = L(y_n + y_p)$$
$$= L(y_n) + L(y_p)$$
$$= 0 + R(x) = R(x).$$

$\Rightarrow y_n + y_p$ is a soln of $L(y) = R(x)$.

$\|^{ly}$,

suppose y_g is a soln of $L(y) = R(x)$ and y_p is a particular soln of $L(y) = R(x)$

Then $L(y_g - y_p) = L(y_g) - L(y_p)$
$$= R(x) - R(x)$$
$$= 0.$$

$\Rightarrow y_g - y_p$ is a soln of $L(y) = 0$.

But we know that y_n is a soln of $L(y) = 0$. So,

$$y_n = y_g - y_p.$$
$$\therefore y_g = y_n + y_p$$

Lemma

If $\phi_1(x)$, $\phi_2(x)$, are the soln. of

$$y'' + P(x).y' + Q(x).y = 0,$$

then the wronskian $W(\phi_1, \phi_2)$ is either zero or never be zero.

Proof:

Given that

ϕ_1 & ϕ_2 are solns of $L(y) = 0$.

$$\Rightarrow \phi_1'' + p\phi_1' + Q.\phi_1 = 0 \quad \text{—— } (1)$$
$$\phi_2'' + p.\phi_2' + Q.\phi_2 = 0 \quad \text{—— } (2)$$

We have,

$$W(\phi_1, \phi_2) = \phi_1 \phi_2' - \phi_2 \phi_1'$$

$$W'(\phi_1, \phi_2) = \phi_1 \phi_2'' + \phi_1' \phi_2' - \phi_1' \phi_2' - \phi_1'' \phi_2$$
$$= \phi_1 \phi_2'' - \phi_1'' \phi_2$$

So, ϕ_1 (2) $-$ ϕ_2 (1)

$$\Rightarrow \phi_1(\phi_2'' + P.\phi_2' + Q.\phi_2) - \phi_2(\phi_1'' + P.\phi_1' + Q.\phi_1) = 0$$

$$\Rightarrow \phi_1 \phi_2'' - \phi_1'' \phi_2 + P(\phi_1 \phi_2' - \phi_1' \phi_2) = 0$$
$$\Rightarrow W' + PW = 0$$
$$\Rightarrow W(x) = W(x_0)\, e^{-\int_{x_0}^{x} P.dx}$$

$$W(x) = 0 \iff W(x_0) = 0.$$

hence the lemma.

Reduction of the order

Theorem :

If ϕ_1 is a soln of

$$y'' + p(x).y' + Q(x)y = 0 \quad \text{—— } (1)$$

and $\phi_1(x) \neq 0$ for any $x \in I$.

Then, the transformation,

$$\phi_2 = v\phi_1 \qquad \{v \text{ is a func. of } x\}$$

reduces (1) to the first order eqn.,

$$\frac{du}{dx} + \left[\frac{2\phi_1' + P.\phi_1}{\phi_1}\right] u = 0$$

where $u = v'$, and the second soln. of (1) is given by,

$$\phi_2(x) = \phi_1(x) \int \frac{1}{\phi_1^2} . e^{-\int P.dx} . dx$$

Proof :

Let ϕ_1 be a soln of $L(y) = 0$ —(1).

Suppose that $\phi_2 = \phi_1 v$ is a soln of (1).

$$\phi_2' = \phi_1 v' + \phi_1' v$$
$$\phi_2'' = \phi_1 v'' + 2\phi_1' v' + \phi_1'' v$$

ϕ_2 is a sdn of ① $\Rightarrow \phi_2'' + P.\phi_2' + Q.\phi_2 = 0.$

$$\Rightarrow (\phi_1 v'' + 2\phi_1' v' + \phi'' v) + P(\phi_1 v' + \phi_1' v) + Q\phi_1 v = 0$$

$$\phi_1 v'' + (2\phi_1' + P\phi_1)v' + (\phi_1'' + P\phi_1' + Q\phi_1)v = 0.$$

$$\Rightarrow \phi_1 v'' + (2\phi' + P\phi_1)v' = 0 \quad \text{—— ②}$$

$$\Rightarrow \frac{v''}{v'} = -\left[\frac{2\phi_1' + P\phi_1}{\phi_1}\right]$$

$$= -\frac{2\phi_1'}{\phi_1} - P$$

$$\Rightarrow \ln v' = -2\ln \phi_1 - \int P.dx$$

$$\Rightarrow v' = \frac{1}{\phi_1^2} e^{-\int P.dx}$$

$$\Rightarrow v = \int \frac{1}{\phi_1^2} e^{-\int P.dx} dx$$

Note that from ②, if we set $v' = u$,

$$\phi_1 u' + (2\phi_1' + P\phi_1)u = 0$$

$$\Rightarrow \frac{du}{dx} + \frac{(2\phi_1 + P\phi_1)u}{\phi_1} = 0$$

Jan 29, Lect - 10

Problem :

Solve.

1. $x^2y'' + xy' - y = 0$, if $\phi_1 = x$ is one of the sdn.
2. $(1-x^2)y'' - 2x.y' + 2y = 0$

1. $x^2y'' + xy' - y = 0$

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$$

$$P = 1/x, \quad Q = -1/x^2$$

$$v = \int \frac{1}{\phi_1^2} e^{-\int P.dx} . dx$$

$$= \int \frac{1}{x^2} e^{-\int \frac{1}{x} dx} dx$$

$$= \int \frac{1}{x^3} dx = \frac{-1}{2} \cdot \frac{1}{x^2}$$

$$\Rightarrow \phi_2 = \phi_1 . v = x . \frac{1}{-2x^2} = \frac{-1}{2x}$$

The solution,

$$\phi = c_1\phi_1 + c_2\phi_2$$
$$= c_1 x - \frac{c_2}{2x}$$

2. By inspection, $y = x$ $\{\phi_1 = x\}$ is a solution.

$$\phi_2 = \phi_1 v.$$

$$P = \frac{-2x}{1-x^2}, \quad Q = \frac{2}{1-x^2}$$

$$v = \int \frac{1}{\phi_1^2} e^{-\int P dx} dx$$

$$= \int \frac{1}{x^2} . e^{-\log(1-x^2)} dx$$

$$= \int \frac{1}{x^2(1-x^2)} dx$$

$$= \int \left(\frac{1}{x^2} + \frac{1}{1-x^2}\right) dx$$

$$\frac{1}{1+x} + \frac{1}{1-x}$$

$$= \frac{-1}{x} + \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$$

Theorem :

Let ϕ_1 be a non-trivial solution of

$$L(y) = a_0(x).y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = 0 \quad \text{—— ①}$$

$x \in I$ $a_0(x) \neq 0$ for any $x \in I$, and $a_0, a_1, \dots, a_n$ are continuous functions on I.

Then the transformation $\phi = \phi_1 v$ reduces the equation into $(n-1)^{th}$ order equation.

$$a_0\phi_1 u^{n-1} + (na_0\phi_1' + a_1\phi_1)u^{n-2} + \dots$$
$$\dots + (na_0\phi_1^{n-1} + \dots + a_{n-1}\phi_1)u = 0 \quad \text{—— (2)}$$

when $u = v'$.

If $u_2, u_3, \dots, u_n$ are $(n-1)$ LI solutions of (2) and if $u_k = v_k'$ for $k \in [2, n] \subset \mathbb{N}$ then,

$\phi_1, v_2\phi_1, \dots, v_n\phi_1$ is a basis of solution of (1), where $v_k = \int_{x_0}^{x} u_k \, dx$

Proof:

let $\phi = \phi_1 v$

$$\phi_1' = \phi_1 v' + \phi_1' v$$
$$\phi_1'' = \phi_1 v'' + 2\phi_1' v' + \phi'' v$$
$$\vdots$$
$$\phi_1^n = \phi_1 v^n + n\phi_1' v^{n-1} + \frac{n(n-1)}{2}\phi_1'' v^{n-2} + \dots + \phi_1^n v$$

Suppose ϕ is a soln of (1), then
$$a_0\phi^n + a_1\phi^{n-1} + \dots + a_n\phi = 0$$

$$\Rightarrow a_0[\phi_1 v^n + n\phi_1' n^{n-1} + \dots + \phi_1^n v] + a_1[\phi_1 v^{n-1} + (n-1)\phi_1' n^{n-2} + \dots + \phi_1^{n-1} v] + \vdots + a_{n-1}[\phi_1 v' + \phi_1' v] + a_n[\phi_1 v] = 0$$

Sorting according to v.

$$\Rightarrow v^n[a_0\phi_1] + v^{n-1}[na_0\phi_1' + a_1\phi_1] + \vdots + v'[na_0\phi^{n-1} + \dots + a_{n-1}\phi_1] + v[a_0\phi^n + a_1\phi^{n-1} + \dots + a_n\phi] = 0$$

$$\Rightarrow a_0\phi_1 v^n + (na_0\phi_1' + a_1\phi_1)v^{n-1} + \dots + (na_0\phi_1^{n-1} + \dots + a_{n-1}\phi_1)v' = 0$$

$$\Rightarrow a_0\phi_1 u^{n-1} + \dots + (na_0\phi_1^{n-1} + \dots + a_{n-1}\phi_1)u = 0,$$

when $u = v'$

This is an $(n-1)^{th}$ order DE and it has $(n-1)$ LI solutions, $\{u_2, u_3, \dots, u_n\}$

Since
$$u_k = v_k', \quad v_k = \int_{x_0}^{x} u_k \, dx.$$

Therefore $\phi_1, \phi_1 v_1, \phi_1 v_2, \dots, \phi_1 v_n$ are the soln of (1).

It remains to show that the above soln is a basis.

Suppose $c_1\phi_1 + c_2\phi_1 v_2 + \dots + c_n\phi_1 v_n = 0$

$$\Rightarrow c_1 + c_2 v_2 + \dots + c_n v_n = 0 \quad \left\{\times \frac{d}{dx}\right\}$$
$$\Rightarrow c_2 v_2' + \dots + c_n v_n' = 0$$

$$\Rightarrow c_2 u_2 + \dots + c_n u_n = 0.$$

$$\Rightarrow c_i = 0, \quad i \in [2, n] \subset \mathbb{N},$$

since u_i are LI.

$\Rightarrow c_1 = 0$, and hence, $c_i = 0 \ \forall i$

$\Rightarrow \phi_1, \phi_1 v_2, \dots, \phi_1 v_n$ are LI and have a basis

and if $u_k = v_k'$ for $k \in [2, n] \subset \mathbb{N}$ then,

$\phi_1, v_2\phi_1, v_3\phi_1, \dots, v_n\phi_1$ is a basis of solution of (1), where
$$v_k = \int_{x_0}^{x} u_k \, dx$$

Method of Variation of Parameter:

Theorem:

let

$$L(y) = y'' + P(x)y' + Q(x).y = R(x) \quad \text{--- (1)}$$

where $P(x)$, $Q(x)$ and $R(x)$ are cont. functions of x defined on an interval I. If ϕ_1, ϕ_2 are two LI soln corresponding to the homog. eqn,

$$y'' + P(x)y' + Q(x)y = 0.$$

then, a particular soln. of (1) is given by,

$$y_p = \phi_1 v_1 + \phi_2 v_2$$

where $v_1 = -\int \frac{R\phi_2}{W}.dx$ and

$$v_2 = \int \frac{R\phi_1}{W} dx$$

where W is the wronskian of $\phi_1, \phi_2 \Rightarrow W(\phi_1, \phi_2)$.

Missed Lectures:

29/01/26 - extra class

02/02/26 - monday

Borrowed notes from a fren.
(attached here)

$a_0\,[\phi_1 (v)^{(n)} + n\phi_1' v^{(n-1)} + \cdots + \phi_1^{(n)} v] + a_1\,[\phi_1 v^{(n-1)} + (n-1)\phi_1' v^{(n-2)} + \cdots + \phi_1^{(n-1)} v] + \cdots +$

$a_{n-1}(\phi_1 v' + \phi_1' v) + a_n(\phi_1 v) = 0$

$\Rightarrow a_0\phi_1 v^{(n)} + (na_0\phi_1' + a_1\phi_1)v^{(n-1)} + \cdots + (na_0\phi_1^{(n-1)} + \cdots + a_{n-1}\phi_1)v' + \underbrace{(a_0\phi_1^{(n)} + a_1\phi_1^{(n-1)} + \cdots + a_n\phi_1)}_{0}v = 0.$

$\Rightarrow a_0\phi_1 v^{(n)} + (na_0\phi_1' + a_1\phi_1)v^{(n-1)} + \cdots + (na_0\phi_1^{(n-1)} + \cdots + a_{n-1}\phi_1)v' = 0$

$\Rightarrow a_0\phi_1 u^{(n-1)} + \cdots + (na_0\phi_1^{(n-1)} + \cdots + a_{n-1}\phi_1)u = 0$ where $u = v'$

This is an $(n-1)^{th}$ order d.e and it has $n-1$ lin-Ind sol^n namely $u_2, u_3, \ldots, u_n$.

Since $u_k = V_k' \Rightarrow V_k = \int_{x_0}^{x_k} u_k\,dx$.

Therefore, $\phi_1, \phi_1 v_2, \phi_1 v_3, \ldots, \phi_1 v_n$ are sol^{ns} of (1), it remains to show that the above sol^n is a basis.

— Suppose $c_1\phi_1 + c_2\phi_1 v_2 + \cdots + c_n\phi_1 v_n = 0$

$\Rightarrow c_1 + c_2 v_2 + \cdots + c_n v_n = 0 \quad \because \phi_1 \neq 0$

$\Rightarrow c_2 v_2' + c_3 v_3' + \cdots + c_n v_n' = 0$ (diff obs)

$\Rightarrow c_2 u_2 + c_3 u_3 + \cdots + c_n u_n = 0$

$\Rightarrow c_i = 0, \; i = 2, \ldots, n$ since u_i's are L-I

$\Rightarrow c_1 = 0$ & hence $c_i = 0\ \forall i \Rightarrow \phi_1, \phi_1 v_2, \ldots, \phi_1 v_n$ are L-I & hence a basis.

Pd.

Method of Variation of Parameter:

Thm: Let $\boxed{L(y) = y'' + P(x)y' + Q(x)\,y = R(x)} \to (1)$ where $P(x)$, $Q(x)$ & $R(x)$ are cont. f^{ns} of x defined on an interval I. If ϕ_1, ϕ_2 are 2 lin. ind sol^{ns} corresponding to the homogeneous eq^n $y'' + P(x)y' + Q(x)y = 0$. Then, a particular sol^n of (1) is given by $\boxed{y_p = \phi_1 v_1 + \phi_2 v_2}$ where $\boxed{v_1 = -\int \frac{R\phi_2}{W}dx}$ and $\boxed{v_2 = \int \frac{R\phi_1}{W}dx}$

where W is Wronskian of ϕ_1, ϕ_2.

29/1/26

$y_p = v_1\phi_1 + v_2\phi_2$

Proof: Let ϕ_1 & ϕ_2 be sol^n of $y'' + p(x)y' + Q(x)y = 0$. $\to (1)$

Assume that $y_p = v_1\phi_1 + v_2\phi_2$ is a sol^n of $y'' + p(x)y' + Q(x)y = R(x)$ $\to (2)$ for some v_1, v_2, f^n of x.

$y_p' = v_1\phi_1' + v_2\phi_2' + v_1'\phi_1 + v_2'\phi_2$

we further assume that $v_1'\phi_1 + v_2'\phi_2 = 0$.

Then, $y_p' = v_1\phi_1' + v_2\phi_2'$

$$y_p'' = v_1\phi_1'' + v_2\phi_2'' + v_1'\phi_1' + v_2'\phi_2'$$

Since y_p is a sol^n of (2),

$$y_p'' + Py_p' + Qy_p = R(x)$$

Therefore, $(v_1\phi_1'' + v_2\phi_2'' + v_1'\phi_1' + v_2'\phi_2') + P(v_1\phi_1' + v_2\phi_2') + Q(v_1\phi_1 + v_2\phi_2) = R(x)$

$$\Rightarrow v_1\underbrace{(\phi_1'' + P\phi_1' + Q\phi_1)}_{\to 0} + v_2\underbrace{(\phi_2'' + P\phi_2' + Q\phi_2)}_{\to 0} + (v_1'\phi_1' + v_2'\phi_2') = R(x)$$

$\because$ ϕ_1, ϕ_2 is a sol^n

$$\Rightarrow v_1'\phi_1' + v_2'\phi_2' = R(x)$$

Thus we have, $v_1'\phi_1 + v_2'\phi_2 = 0$ & $v_1'\phi_1' + v_2'\phi_2' = R(x)$

Solving for v_1' & v_2',

$$v_1' = \frac{-R(x)\phi_2}{W}, \quad \underline{v_1 = \int \frac{-R\phi_2}{W}} \qquad \Big| \; W = \phi_1\phi_2' - \phi_1'\phi_2$$

$$v_2' = \frac{R(x)\phi_1}{W}, \quad \underline{v_2 = \int \frac{R\phi_1}{W}}$$

Bad.

* For solving $y'' + P(x)y' + Q(x)y = R(x)$,

» Step 1: Find two sol^n ϕ_1, ϕ_2 of $y'' + p(x)y' + Q(x)y = 0$

» Step 2: Find a particular sol^n $y_p = v_1\phi_1 + v_2\phi_2$ of $y'' + p(x)y' + Q(x)y$

» Step 3: The gen. sol^n $y_g = y_h + y_p \Rightarrow \boxed{y_g = c_1\phi_1 + c_2\phi_2 + v_1\phi_1 + v_2\phi_2}$

$$v_1 = -\int \frac{R\phi_2}{W}dx \quad \& \quad v_2 = \int \frac{R\phi_1}{W}dx$$

(Q) 1 Find a particular sol^n of (1) $y'' - \frac{2y}{x^2} = 2x$

(2) $y'' - 2y' + 3y = 64xe^{x}$

H.W. Solve $y'' + y = \sec x$, $y'' + 4y = \tan 2x$.

(Q), Ans (1) $y'' - \frac{2y}{x^2} = x$

$$P = 0 \qquad Q = \frac{-2}{x^2} \qquad R = 2x$$

Consider $y'' - \frac{2y}{x^2} = 0$

by inspection, $\boxed{\phi_1 = x^2}$ is a sol^n.

second solⁿ, $\phi_2 = \phi_1 v$

$$v = \int \frac{1}{\phi_1^2} e^{-\int P dx} dx = \int \frac{1}{x^4} dx = \frac{x^{-4+1}}{-4+1} = \frac{-1}{3x^3}$$

$$\boxed{\phi_2 = \frac{-1}{3x}}$$

$$\therefore \boxed{\phi_1' = 2x} \quad \boxed{\phi_2' = \frac{1}{3x^2}}$$

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} x^2 & -1/3x \\ 2x & 1/3x^2 \end{vmatrix} = \frac{1}{3} + \frac{2}{3} = 1$$

Step 2: To find PS, $y_p = v_1\phi_1 + v_2\phi_2$

$$v_1 = -\int \frac{R\phi_2}{W} dx = -\int \frac{x(-1/3x)}{1} dx = \int \frac{1}{3} dx = \frac{x}{3}$$

$$v_2 = \int \frac{R\phi_1}{W} dx = \int \frac{x(x^2)}{1} dx = \int x^3 dx = \frac{x^4}{4}$$

$$y_p = v_1\phi_1 + v_2\phi_2 = \left(\frac{x}{3}\right)(x^2) + \left(\frac{x^4}{4}\right)\left(\frac{-1}{3x}\right)$$

$$= \frac{4}{4}\frac{x^3}{3} - \frac{x^3}{12} = \frac{3x^3}{12} = \frac{x^3}{4}$$

Q) $y'' - 2y' - 3y = 64xe^x$

A) $P = -2,\ Q = -3 \quad R = 64xe^x$

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0 \rightarrow r = \dfrac{2 \pm \sqrt{4+12}}{2} = \dfrac{2 \pm 4}{2} = 1 \pm 2 = -1, 3$

$$\boxed{\phi_1 = e^{-x} \quad \phi_2 = e^{3x}}$$

$$\boxed{\phi_1' = -e^{-x}, \quad \phi_2' = 3e^{3x}}$$

$$W = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 3e^{2x} + e^{2x} = 4e^{2x}$$

P.S, $y_p = v_1\phi_1 + v_2\phi_2$

$$v_1 = -\int \frac{R\phi_2}{W} dx = -\int \frac{64xe^x \cdot e^{3x}}{4e^{2x}} dx = -48\int xe^{2x} dx$$

$$= -48\left[x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} dx\right] = -48\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]$$

$$= -24xe^{2x} + 12e^{2x}$$

$v_2 =$

→ Theorem: Let $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = b(x)$ where $a_i(x)$, $b(x)$ are cont f^n on I. If $\phi_1, \phi_2, \dots, \phi_n$ → a set of n L.I soln of the corresponding homogeneous eqn, then a particular soln of $L(y) = b(x)$ is given by $y_p = v_1\phi_1 + v_2\phi_2 + \dots + v_n\phi_n$ where $v_k(x)$ is defined by $v_k(x) = \int_{x_0}^{x} \frac{w_k(x)}{w(x)} b(x)\, dx$ where $w(x)$ is the wronskian of $\phi_1, \phi_2, \dots, \phi_n$ and $w_k(x)$ is the det obtained from w by replacing column $\begin{bmatrix} \phi_k \\ \phi_k' \\ \vdots \\ \phi_k^{(n-1)} \end{bmatrix}$ by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Proof:

Let $\phi_1, \phi_2, \dots, \phi_n$ be the soln of the homo. part and suppose $y_p = v_1\phi_1 + v_2\phi_2 + \dots + v_n\phi_n$.

Consider $y_p' = v_1\phi_1' + v_2\phi_2' + \dots + v_n\phi_n' + v_1'\phi_1 + v_2'\phi_2 + \dots + v_n'\phi_n$

We assume that $v_1'\phi_1 + \dots + v_n'\phi_n = 0$.

Then, $y_p' = v_1\phi_1' + v_2\phi_2' + \dots + v_n\phi_n'$

$y_p'' = v_1\phi_1'' + v_2\phi_2'' + \dots + v_n\phi_n'' + v_1'\phi_1' + \dots + v_n'\phi_n'$

Again assume $v_1'\phi_1' + \dots + v_n'\phi_n' = 0$

Proceed like this, we have,

$$\left.\begin{aligned} & v_1'\phi_1 + \dots + v_n'\phi_n = 0 \\ & v_1'\phi_1' + v_2'\phi_2' + \dots + v_n'\phi_n' = 0 \\ & \vdots \\ & v_1'\phi_1^{n-1} + v_2'\phi_2^{n-1} + \dots + v_n'\phi_n^{n-1} = b(x) \\ & \cancel{v_1\phi_1^{n} + v_2\phi_2^{n} + \dots + v_n\phi_n^{n} = b(x)} \end{aligned}\right\} (A)$$

Accordingly, we have,

$y_p = v_1\phi_1 + \dots + v_n\phi_n$

$y_p' = v_1\phi_1' + \dots + v_n\phi_n'$

$y_p^{n-1} = v_1\phi_1^{n-1} + \dots + v_n\phi_n^{n-1}$

$\vdots$

$y_p^{n} = v_1\phi_1^{n} + \dots + v_n\phi_n^{n} + \underbrace{v_1'\phi_1^{(n-1)} + \dots + v_n'\phi_n^{n-1}}_{b(x)}$

$= v_1\phi_1^{n} + \dots + v_n\phi_n^{(n)} + b(x)$

$L(y_p) = v_1 L(\phi_1) + \dots + v_n L(\phi_n) + b = b(x)$

From (A),

we need to solve for $v_1', v_2', \dots, v_n'$

$$v_1' = \begin{vmatrix} 0 & \phi_2 & \cdots & \phi_n \\ 0 & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ b(x) & \phi_2^{n-1} & \cdots & \phi_n^{n-1} \end{vmatrix} \Bigg/ \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \cdots & \phi_n^{n-1} \end{vmatrix} = \frac{b(x) \begin{vmatrix} 0 & \phi_2 & \cdots & \phi_n \\ 0 & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ 1 & \phi_2^{n-1} & \cdots & \phi_n^{n-1} \end{vmatrix}}{w}$$

$= \frac{b(x)\, W_1(x)}{W} \Rightarrow v_1 = \int \frac{W_1}{W} b(x)\,dx$

2/2/26

Recap:

Var^n of Parameters: $\to y'' + Py' + Qy = R$

$\to a y'' + a_1 y' + a_2 y = b$

$v_1 = \int \frac{-b\phi_2\,dx}{a_0 W} \qquad v_2 = \int \frac{b\phi_1}{a_0 W}\,dx$

$\to a_0 y^{(n)} + a_1 y^{n-1} + \dots + a_n y = b$

$v_k = \int_{x_0}^{x} \frac{W_k(s)\, b(s)}{a_0\, W(s)}\,ds$

$\to y'' + Py' + Qy = R$, ϕ_1, ϕ_2 sol^n of Hom-DE

$W(\phi_1, \phi_2)(x) = W(\phi_1, \phi_2)(x_0)\, e^{-\int P\,dx}$

$a_0 y'' + a_1 y' + a_2 y = 0$

$W(\phi_1, \phi_2)(x) = W(\phi_1, \phi_2)(x_0)\, e^{-\int_{x_0}^{x} \frac{a_1(s)}{a_0(s)}\,ds}$

Thm: Let $\phi_1, \phi_2, \dots, \phi_n$ be n L.I sol^{ns} of $L(y) := a_0(x) y^{(n)} + \dots + a_n(x) y = 0$ where $a_0(x) \neq 0$, $\forall x \in I$ & $a_0, \dots, a_n$ are cont. f^{ns} of x on I & let $x_0 \in I$

Then, $W(\phi_1, \phi_2, \dots, \phi_n)(x) = W(\phi_1, \phi_2, \dots, \phi_n)(x_0)\, e^{-\int_{x_0}^{x} \frac{a_1(s)}{a_0(s)}\,ds}$

Pf: We have $W = \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \dots & \phi_n^{n-1} \end{vmatrix}$

$$W' = \begin{vmatrix} \phi_1' & \phi_2' & \dots & \phi_n' \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \dots & \phi_n^{n-1} \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1'' & \phi_2'' & \dots & \phi_n'' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \dots & \phi_n^{n-1} \end{vmatrix} + \dots + \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n} & \phi_2^{n} & \dots & \phi_n^{n} \end{vmatrix}$$

$$= 0 + 0 + \dots + \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & & \phi_n^{n-2} \\ \phi_1^{n} & \phi_2^{n} & \dots & \phi_n^{n} \end{vmatrix}$$

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & \dots & \phi_n^{n-2} \\ \phi_1^{n} & \phi_2^{n} & \dots & \phi_n^{n} \end{vmatrix} = \frac{1}{a_0} \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & \dots & \phi_n^{n-2} \\ a_0\phi_1^{n} & a_0\phi_2^{n} & \dots & a_0\phi_n^{n} \end{vmatrix}$$

$$= \frac{1}{a_0}\begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & \cdots & \phi_n^{n-2} \\ -a_1\phi_1^{n-1}-a_2\phi_1^{n-2}-a_n\phi_n & -a_1\phi_2^{n-1}-a_n\phi_2 & \cdots & -a_1\phi_n^{n-1}-\cdots-a_n\phi_n \end{vmatrix}$$

$R_n \to R_n + a_n R_1 + a_{n-1}R_2 + \cdots + a_2 R_{n-1}$

$$w' = \frac{1}{a_0}\begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & \vdots & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & & \phi_n^{n-2} \\ -a_1\phi_1^{n-1} & -a_1\phi_2^{n-1} & \cdots & -a_1\phi_n^{n-1} \end{vmatrix}$$

$$= -\frac{a_1}{a_0} w$$

$$w' + \frac{a_1}{a_0} w = 0 \implies w = c\, e^{-\int_{x_0}^{x} \frac{a_1}{a_0} dx}$$

$$\implies w(\phi_1,\ldots,\phi_n)(x) = w(\phi_1,\ldots,\phi_n)(x_0)\, e^{-\int_{x_0}^{x} \frac{a_1}{a_0} dx}$$

- <u>Existence & Uniqueness of IVP of 1st order eq^n:</u>

$$\left.\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right\} (1)$$

- <u>Thm</u>: Let $f(x,y)$ be cont. in the domain D. Then, any function ϕ is a sol^n of the IVP $y' = f(x,y)$; $y(x_0) = y_0$ on I.

$\Leftrightarrow$ it is a sol^n of the integral eq^n $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \longrightarrow (2)$.

<u>Pf</u>: Let ϕ be a sol^n of IVP, (1)

$$\phi' = f(x,\phi) \longrightarrow (3)$$

$$\phi(x_0) = y_0$$

Since ϕ_1 is diff, ϕ is cont as well.

f is cont & ϕ is cont $\Rightarrow f$ is integrable.

$$\Rightarrow \int_{x_0}^{x} \phi'\, dx = \int_{x_0}^{x} f(x,\phi)\, dx$$

$$\Rightarrow \phi(x) - \phi(x_0) = \int_{x_0}^{x} f(t,\phi(t))\, dt$$

$$\Rightarrow \phi(x) = y_0 + \int_{x_0}^{x} f(t,\phi(t))\, dt$$

$\Rightarrow \phi$ <u>is a sol^n of integral</u> (2)

Conversely, suppose ϕ is a sol^n of (2),

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t,\phi(t))\, dt.$$

Note that $\phi(x_0) = y_0$ & $\phi'(x) = f(x, \phi(x))$

$\Rightarrow \phi$ is a solⁿ of (1).

Lipschitz condition:

Defⁿ: A function $f(x,y)$ defined on a domain $D \subseteq \mathbb{R}^2$ is said to satisfy the Lipschitz condition with respect to y, if $\exists$ constt K s.t

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$$

where (x,y_1) & $(x,y_2) \in D$

. We also say $f \in Lip(D,K)$, K: Lipschitz constt.

$f(x,y) = xy$, $D = \{(x,y) : |x| \leq 1, |y| \leq 1\}$

$|f(x,y_1) - f(x,y_2)| = |xy_1 - xy_2| \leq |x||y_1 - y_2| \leq 1\,|y_1 - y_2|$

$\Rightarrow f \in Lip(D,1)$

$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$

$$\left|\frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2}\right| \leq K.$$

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4/2/26

Thm: Let $f(x,y)$ be cont. fⁿ defined over a rectangle $R = \{(x,y) : |x - x_0| \leq a, |y - y_0| \leq b\}$ where $a, b > 0$. If $\frac{\partial f}{\partial y}$ exists and cont on R, then $f(x,y)$ satisfies the Lipschitz condition wrt y in R & Lip. constt K is given by $K = \underset{(x,y)\in R}{\text{lub}} |f_y(x,y)|$

Pf: Since $\frac{\partial f}{\partial y}$ is cont on a closed rectangle, it is bdd.

$\Rightarrow$ supremum exists.

Let $K = \underset{(x,y)\in R}{\text{lub}} \left|\frac{\partial f}{\partial y}\right|$

Let (x,y_1) & $(x,y_2) \in R$

Then, by FTDC, $f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(y_1 - y_2)$

$\Rightarrow |f(x,y_1) - f(x,y_2)| \leq \left|\frac{\partial f}{\partial y}\right| |y_1 - y_2| \leq K|y_1 - y_2|$

Pd

↳ Converse need not be true

Ex : $f(x,y) = x^2|y|$ in $D = \{|x| \leq 1, |y| \leq 1\}$

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Theorem :

Let $f(x,y)$ be a continuous fn defined over a rectangle

$$R = \{(x,y) : |x-x_0| \le a, \ |y-y_0| \le b\}$$

where $a, b > 0$. If $\frac{\partial f}{\partial y}$ exists and continuous on R, then $f(x,y)$ satisfies the Lipschitz condition with respect to y in R and the Lip. constant K is given by

$$K = \underset{(x,y)\in R}{\text{lub}} |f_y(x,x)|$$

Proof :

Since $\frac{\partial f}{\partial y}$ is continuous on a closed rect., it is bounded $\Rightarrow$ supremum exist.

Let,

$$K = \underset{(x,y)\in \mathbb{R}}{\text{lub}} \left|\frac{\partial f}{\partial y}\right|$$

Let (x, y_1) and $(x, y_2) \in R$, then by FTDC

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(y_1 - y_2)$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le \left|\frac{\partial f}{\partial y}\right| |y_1 - y_2|$$

$$\le K|y_1 - y_2|$$

Converse need not be true.

$$f(x,y) = x^2|y| \quad \text{in} \quad D = \{|x| \le 1, \ |y| \le 1\}$$

Problems :

Check whether the following funcs satisfy L.C. and if so, find k.

1. $f(x,y) = y^{1/2}, \ |x| \le 1, \ 0 \le y \le 2$
2. $f(x,y) = xy^2, \ D = \{|x| \le 1, \ |y| \le 1\}$
3. $f(x,y) = (y+y^2)\frac{\cos x}{x^2}, \ D = \{|y| \le 1, \ |x-1| < \frac{1}{2}\}$

Gronwall Inequality

Let $f(x)$ and $g(x)$ be two non-negative continuous func. for $x \ge x_0$. Let K be any non-negative constant. Then the inequality,

$$f(x) \le K + \int_{x_0}^{x} g(t).f(t).dt, \quad x \ge x_0$$

$$\Rightarrow f(x) \le K e^{\int_{x_0}^{x} g(t).dt}, \quad x \ge x_0$$

Proof :

Given that f and g are cont. and

$$f(x) \le K + \int_{x_0}^{x} g(t).f(t).dt \quad \text{——— ①}$$

Let $F(x) = K + \int_{x_0}^{x} g(t).f(t).dt$

$$F(x_0) = K \ne 0$$

$$F'(x) = g(x).f(x)$$

We have from ①,

$$f(x) \le F(x).$$

$$\Rightarrow \frac{f(x)}{F(x)} \le 1$$

$$\frac{g(x).f(x)}{F(x)} \le g(x)$$

$$\Rightarrow \frac{F'(x)}{F(x)} \le g(x)$$

$$\Rightarrow \int_{x_0}^{x} \frac{F'(x)}{F(x)} dx \le \int_{x_0}^{x} g(x).dx$$

$$\Rightarrow \ln(F(x))\Big|_{x_0}^{x} \le \int_{x_0}^{x} g(x)\,dx$$

$$\ln(F(x)) - \ln K \le \int_{x_0}^{x} g(x).dx$$

$$\Rightarrow F(x) \le K.e^{\int_{x_0}^{x} g(x).dx}$$

$$\Rightarrow K + \int g(t).f(t).dt \le K.e^{\int g(t).dx}$$

$$\Rightarrow f(x) \le F(x) \le K.e^{\int g(t).dt}$$

Hence the result.

Corollary :

If

$$f(x) \le K \int_{x_0}^{x} f(t)\,dt \quad \forall\, x \ge x_0$$

Then $f(x) \equiv 0 \quad \forall \; x \ge x$

Proof :

Given that

$$f(x) \le K \int_{x_0}^{x} f(t)\,dt$$

$$\le \varepsilon + \int_{x_0}^{x} K.f(t)\,dt, \quad \varepsilon > 0.$$

By Grimwell inequality,

$$f(x) \le \varepsilon . e^{\int_{x_0}^{x} K.dt} = \varepsilon e^{K(x-x_0)}$$

Since ε is arbitrary, we have $f(x) = 0$.

$$y' = f(x, y)$$

$$y(x_0) = y_0 \approx y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

Picard's Approximation Scheme

Start with initial approx. $\phi_0(x_0) = y_0$ and successively find the approx ϕ_K of ϕ as follows.

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\,dt$$

$$\vdots$$

$$\phi_K(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{K-1}(t))\,dt$$

If $\phi_K \to \phi$, then ϕ will be a soln of I. eqn.

$$\phi = \lim_{K\to\infty} \phi_K(x) = y_0 + \lim_{K\to\infty} \int f(t, \phi_{K-1}(t))\,dt$$

$$= y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$$

Problems:

Solve,

1. $y' = xy$, $y(0) = 1$, using Picard's approx. scheme.

2. Find the first 4 approx. of $y' = 1 + xy$, $y(0) = 1$

3. Find 4 approx of $y' = x + y$, when $y(0) = 1$.

---

1. $y' = xy$

$y(0) = 1$

we have,

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$= 1 + \int_0^x t.y.dt$$

$$\phi_0 = y_0 = 1$$

$$\phi_1 = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

$$= 1 + \int_0^x t\,dt = 1 + \frac{x^2}{2}$$

$$\phi_2 = y_0 + \int_0^x f(t, \phi_1)\,dt$$

$$= 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right) dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} = 1 + \frac{x^2}{2} + \frac{(x^2/2)^2}{2!}$$

$$\phi_3 = 1 + \frac{(x^2/2)}{1!} + \frac{(x^2/2)^2}{2!} + \frac{(x^2/2)^3}{3!}$$

$$\phi_K \to e^{(x^2/2)}$$

2. $y' = 1 + xy$

$y(0) = 1$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_1 = 1 + \int_0^x (1+t)\,dt$$

$$\phi_1 = y_1 = 1 + x + \frac{x^2}{2}, \quad \phi_0 = 1$$

$$\phi_2 = y_2 = 1 + \int_0^x \left(1+t+t^2+\frac{t^3}{2}\right) dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$\phi_3 = y_3 = 1 + \int_0^x \left(1+t+t^2+\frac{t^3}{2}+\frac{t^4}{3}+\frac{t^5}{8}\right) dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

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Theorem:

The sequence of successive approximation

$$\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(x))\,dt, \quad k = 1, 2, \dots$$

with $\phi_0(x) = y_0$ exists as continuous func. on $I = |x - x_0| < h = \min\{a, \frac{b}{M}\}$ and $(x, \phi_n(x)) \in R$ where,

$$R = \{(x,y) \mid |x - x_0| \le a,\ |y - y_0| \le b,\ a, b > 0\}$$

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- Jahnavi

and $M > 0$ for each $x \in I$ and ϕ_k satisfies

$$|\phi_k(x) - \phi_0(x)| \le M|x - x_0| \quad \forall x \in I.$$

Proof {by induction}

When $k=0$, the result is obvious when $k=1$,

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

ϕ_1 is continuous on a closed rect R and hence it is bounded. $\therefore \exists M > 0$, $|f(t, \phi_0)| \le M$ on I.

$$|\phi_1 - \phi_0| = \left| \int_{x_0}^{x} f(t, \phi_0)\,dt \right|$$

$$\le \int_{x_0}^{x} |f(t, \phi_0)|\,dt$$

$$\le \int_{x_0}^{x} M\,dt = M(x - x_0)$$

$$\le M|x - x_0|$$

Let us assume that the result is true for $k = n$ and we will prove it for $k = n+1$.

We have, $|\phi_n(x) - \phi_0| \le M|x - x_0|$

$(x, \phi_n) \in R$, ϕ_n is continuous.

We have $\phi_{n+1}(x) = \phi_0 + \int_{x_0}^{x} f(t, \phi_n(t))\,dt.$

Since $f(t, \phi_n(t))$ is cont. on R, $\exists\, m > 0$ such that $|f(t, \phi_n(t))| \le M$, $M > 0$.

$$|\phi_{n+1}(x) - \phi_0(x)| = \left| \int_{x_0}^{x} f(t, \phi_n(t))\,dt \right|$$

$$\le \int_{x_0}^{x} |f(t, \phi_n(t))|\,dt$$

$$\le M|x - x_0|$$

Hence the theorem.

Picard's existence & Uniqueness theorem

Let $f(x,y)$ be a continuous real valued func. defined on a closed rectangle,

$$R = \{(x,y),\ |x-x_0| \le a,\ |y-y_0| \le b,\ a,b>0\}$$

and let $|f(x,y)| \le M,\ \forall (x,y) \in R.$

Further, suppose that $f(x,y)$ satisfy the Lipshitz condition,

$$|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2|$$
$$\forall (x,y_1),(x,y_2) \in R$$

with lipshitz constant k.

Then the successive approximation,

$$\phi_0 = y_0,\quad \phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t,\phi_n(t))\,dt$$

converges in the interval,

$$I = |x - x_0| < h = \min\left\{a, \frac{b}{M}\right\}$$

to a unique solution.

$\phi(x)$ of the IVP, $y' = f(x,y)$
$y(x_0) = y_0$ on I.

Proof:

We prove the theorem in three steps.

Step 1: The seq. $\{\phi_n\}$ converges.

Step 2: The lim is a soln of IVP

Step 3: The soln is unique.

Step 1: We have,

$$\phi_0 = y_0,\quad \phi_n = y_0 + \int_{x_0}^{x} f(t,\phi_{n-1}(t))\,dt$$

we can write,

$$\phi_n = \phi_0 + (\phi_1 - \phi_0) + \dots + (\phi_n - \phi_{n-1})$$
$$= \phi_0 + \sum_{k=1}^{n} (\phi_k - \phi_{k-1})$$

$\Rightarrow \phi_n$ is the n^{th} parted sum of the series,

$$\phi_0 + \sum_{1}^{\infty} (\phi_k - \phi_{k-1})$$

If we are able to show that the series converges then, the sequence will converge.

By the proven (?) theorem, we have,

$$|\phi_{n+1}(x) - \phi_0(x)| \le M|x - x_0|$$

without the loss of generality. Let us assume that $x \ge x_0$.

Consider,

$$|\phi_2(x) - \phi_1(x)| \le \int_{x_0}^{x} |f(t,\phi_1(t)) - f(t,\phi_0(t))|\,dt$$
$$\le \int_{x_0}^{x} k\,|\phi_1(t) - \phi_0(t)|\,dt$$
$$= k\int_{x_0}^{x} M|t - x_0|\,dt$$
$$= Mk \cdot \frac{(x-x_0)^2}{2}$$

We claim that,

$$|\phi_n(x) - \phi_{n-1}(x)| \le M\frac{k^{n-1}(x-x_0)^n}{n!}$$

Proof is by induction.

The result is true for $n = 1, 2$.

We assume that it is true for n and will prove for $n+1$.

$$\phi_{n+1}(x) - \phi_n(x) = \int_{x_0}^{x} (f(t,\phi_n(t)) - f(t,\phi_{n-1}(t)))\,dt$$

$$|\phi_{n+1} - \phi_n| \le \int_{x_0}^{x} |f(t,\phi_n) - f(t,\phi_{n-1})|\,dt$$
$$\le \int_{x_0}^{x} k|\phi_n - \phi_{n-1}|\,dt$$
$$\le k\int_{x_0}^{x} \frac{M \cdot k^{n-1}(t-x_0)^n}{n!}\,dt$$
$$= \frac{Mk^n(x-x_0)^{n+1}}{(n+1)!}$$

$$|\phi_{n+1} - \phi_n| \le M \frac{K^n (x-x_0)^{n+1}}{(n+1)!}$$

$$\le \frac{M}{K} \cdot \frac{K^{n+1} \cdot h^{n+1}}{(n+1)!} \quad \{\text{given, } |x-x_0| \le h$$

$$= \frac{M}{K} \frac{(Kh)^{n+1}}{(n+1)!}$$

$$\sum_{1}^{\infty} |\phi_{n+1} - \phi_n| \le \frac{M}{K} \sum \frac{(Kh)^{n+1}}{(n+1)!}$$

RHS is a series of (+ve) terms,

converges to,

$$\frac{M}{K}(e^{kh} - 1).$$

Thus the series $\phi_0 + \Sigma(\phi_k - \phi_{k-1})$ is dominated by a convg. series of +ve terms. Hence by Weistrass - M test, the series $\phi_0 + \Sigma(\phi_k - \phi_{k-1})$ converges uniformly to $\phi(x)$.

Since ϕ_k's are continuous and $\phi_k \longrightarrow \phi$ uniformly, ϕ is continuous too.

Step 2 :

ϕ is a soln of IVP.

$\therefore$ It is enough to prove that ϕ is the soln of the integral eqn,

$$y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt .$$

We claim that $f(t, \phi_n(t))$ converges uniformly to $f(t, \phi_n(t))$.

We know that $\phi_n \longrightarrow \phi$ uniformly, given $\varepsilon > 0$, $\exists$ $n_0 > 0$ such that,

$$|\phi_n(x) - \phi(x)| < \frac{\varepsilon}{K} \quad \forall \; n \ge n_0$$

$$|f(t, \phi_n(t)) - f(t, \phi(t))| \le K|\phi_n(t) - \phi(t)|$$

$$\le K \frac{\varepsilon}{K}$$

$$= \varepsilon, \quad \forall \; n \ge n_0$$

$\Rightarrow f(t, \phi_n) \longrightarrow f(t, \phi)$ uniformly.

$$\therefore \phi(x) = \lim_{n\to\infty} \phi_{n+1}(x)$$

$$= \lim_{n\to\infty} \left[y_0 + \int_{x_0}^{x} f(t, \phi_n(t))\, dt \right]$$

$$= y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$\Rightarrow$ ϕ is the soln of integral eqn and hence it is a soln of IVP.

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Piccards existence and uniqueness

Step 3: Uniqueness of soln.

Let ϕ and ψ be two soln of the IVP.

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt$$

$$\phi(x) - \psi(x) = \int [f(t, \phi(t)) - f(t, \psi(t))]\, dt$$

$$|\phi(x) - \psi(x)| \le \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \psi(t))|\, dt$$

$$\le K \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt$$

$\hookrightarrow$ f is Lipschitz

Hence by the corollary of Gronwall's inequality,

$$|\phi(x) - \psi(x)| \equiv 0 \Rightarrow \phi(x) = \psi(x).$$

ε - approximation soln

Suppose we have the IVP,

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

An ε - approximation soln is a function $y(x)$ on an interval $I = [x_0, x_0+h]$ satisfying the following properties.

1. $(x, y) \in D$
2. $y \in C'$ except possibly for finite points $\{y \in C'(I \setminus S),\ S \subset I\}$
3. $|y' - f(x,y)| < \varepsilon, \quad \forall\ x \in I \setminus S, \quad \varepsilon > 0$

Existence :

Theorem:

Suppose that $f(x,y)$ in ① is continuous on the rectangle,

$$R = \{(x,y) : |x - x_0| \le a,\ |y - y_0| \le b,\ a, b > 0\}$$

Let,

$$M = \max_{(x,y)\in R} |f(x,y)| \quad \& \quad h = \min\left\{a, \frac{b}{M}\right\}$$

Then for given $\varepsilon > 0$, $\exists$ an ε-approx. soln y for the IVP ① on $|x - x_0| \le h$.

Proof:

We will try to construct an ε-apprx. solution for the IVP.

We partition the interval $[x_0, x_0+h]$ into the sub-intervals such that,

$$x_0 < x_1, < x_2 < \dots < x_n = x_0 + h$$

where $|x_i - x_{i-1}| < \min\{\delta, \frac{\delta}{M}\}$, $\delta > 0$

We define

$$y(x) = y(x_{i-1}) + (x - x_{i-1}) f(x_{i-1}, y_{i-1})$$
$$i \in [1, n]$$
$$x \in [x_{i-1}, x_i]$$

Clearly, y is cont. and has piecewise cont derivatives. Hence y satisfies the condition ② of ε- approx.

We claim that $(x, y) \in R$.

It is enough to prove that

$$|y(x_i) - y_0| < b, \quad i \in [1, n]$$

From ③ we have,

$$y(x_1) - y_0 = (x_1 - x_0) f(x_0, y_0)$$
$$y(x_2) - y(x_1) = (x_2 - x_1) f(x_1, y_1)$$
$$\vdots$$
$$y(x_k) - y(x_{k-1}) = (x_k - x_{k-1}) . f(x_{k-1}, y_{k-1})$$

$$\therefore\ y(x_k) - y_0 = \sum_{i=1}^{k} (x_i - x_{i-1}) f(x_{i-1}, y_{i-1})$$

$$|y(x_k) - y_0| \le \sum_{i=1}^{k} (x_i - x_{i-1}) |f'(x_{i-1}, y_{i-1})|$$
$$\le M \sum (x_i - x_0)$$
$$\le b$$

$$\therefore (x, y) \in R.$$

From ②,

$$y(x) - y(x_{i-1}) = (x - x_i) . f(x_{i-1}, y_{i-1})$$
$$|y(x) - y(x_{i-1})| \le |x - x_i| |f(x_{i-1}, y_{i-1})|$$
$$\le M . \frac{\delta}{M} = \delta$$

Since f is cont. in R, it is uniformly continuous.

$$|f(x,y) - f(x_i, y_i)| < \varepsilon \quad \text{where} \quad |x - x_i| < \delta,\ |y - y_i| < \delta$$

for a $(x, y), (x_i, y_i) \in R$

$$|y' - f(x,y)|$$
$$= |f(x_{i-1}, y_{i-1}) - f(x,y)| < \varepsilon$$

$\Rightarrow$ Condition ③ of ε-app satisfies,

s.t. $x_0 < x_1 < x_2 < \ldots\ldots < x_n = x_0 + h.$

when $|x_i - x_{i-1}| < \min\{\delta, \frac{\delta}{M}\}$, $\delta > 0.$

we define,

$$y(x) = y(x_{i-1}) + (x - x_{i-1}) f(x_{i-1}, y_{i-1}) \longrightarrow ②$$

$$i = 1, \ldots, n$$
$$x \in [x_{i-1}, x_i]$$

Feb 11, Lect - 15 ?

Theorem :

The n^{th} approximation $\phi_n(x)$ of $\phi(x)$ of IVP satisfies

$$|\phi_n(x) - \phi(x)| \leq \frac{M}{k} \frac{(kh)^{n+1}}{(n+1)!} e^{kh}, \quad x \in I$$

Proof:

We have $\phi_n(x) = \phi_0 + \sum_{i=1}^{n} (\phi_i - \phi_{i-1})$

$$\phi(x) = \phi_0 + \sum_{1}^{\infty} (\phi_i - \phi_{i-1})$$

$$|\phi_n(x) - \phi(x)| = \left|\sum_{n+1}^{\infty} (\phi_i - \phi_{i-1})\right|$$

$$\leq \sum_{n+1}^{\infty} |\phi_i - \phi_{i-1}|$$

$$\leq \sum_{n+1}^{\infty} \frac{M}{k} \frac{(kh)^i}{i!}$$

$$= \frac{M}{k} \frac{(kh)^{n+1}}{(n+1)!} \sum \frac{(kh)^i}{i!}$$

Cauchy - Peano's theorem

Let $f(x,y)$ be continuous on the rect.

$$R = \{(x,y) : |x - x_0| \leq a, \; |y - y_0| \leq b, \; a, b > 0\}$$

Then $\exists$ a solution to IVP $y' = f(x,y)$ $y(x_0) = y_0$ in the interval

$$|x - x_0| < h = \min\{a, \frac{b}{M}\}$$

$$M = \max_{x \in R} |f(x)|$$

Proof:

Let $\varepsilon_n = 1/n$.

By ε-approx theorem, for each ε_n, $\exists$ an ε_n-approximation $\{y_n\}$ of the IVP such that,

$$|y_n - y_0| \leq b$$

$\Rightarrow |y_n| \leq |y_0| + b$

$\Rightarrow \{y_n\}$ is uniformly bounded.

Moreover,

$$|y_n(x) - y_n(\tilde{x})| \leq M|x - \tilde{x}|, \quad x, \tilde{x} \in I$$

$\Rightarrow \{y_n\}$ is equi-continuous on I.

Then $\{y_n\}$ is uniformly bounded and equi-continuous.

Hence by Arzela-Ascoli's theorem, $\exists \{y_{n_k}\}$ of $\{y_n\}$ which converges to $y(x)$ uniformly.

Hence y is cont. and,

$$|y(x) - y(\tilde{x})| \leq M|x - \tilde{x}|, \quad x, \tilde{x} \in I$$

We claim that limit is a soln of IVP.

we define

$$e_{n_k}(x) = \begin{cases} y_{n_k}' - f(x, y_{n_k}(x)) & \text{if } y_{n_k}' \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

$y_{n_k}' = f(x, y_{n_k}(x)) + e_{n_k}(x)$ except at finite no. of points.

By integrating from $x_0 \to x$, we get

$$y_{n_k}(x) = y_0 + \int_{x_0}^{x} \left[f(x, y_{n_k}(x)) + e_{n_k}(x) \right] dx \longrightarrow *$$

We know that

$$y_{n_k} \to y(x) \text{ uniformly.}$$

Then $f(x, y_{n_k}(x)) \longrightarrow f(x,y)$ uniformly.

Again, $|e_{n_k}(x)| \leq \varepsilon_n = \frac{1}{n_k} \to 0$ as $n \to \infty$.

Hence for $(*)$ we get

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$\Rightarrow$ y is a soln of the IVP

Feb 12, Lect - 18 ?

Banach fixed point theorem :

Suppose (X, d) is a complete Metric space and $T: X \longrightarrow X$ is a contradiction, then T has a unique fixed point $x^* \in X$.

Corollary :

Let $T: X \longrightarrow X$ be such that T^k is a contradiction for some $k \geq 1$. Then, T has a unique fixed point.

Existence of Uniqueness of IVP by Fixed Point Theorem:

Under the same hypothesis of Picard's theorem, IVP has a unique soln.

Proof :

Let $X = \{ y \in C[x_0, x_0 + h] \text{ s.t. } |y - y_0| \leq b \}$

X is a closed ball in a Banach space, $C[x_0, x_0+h]$ with supremum norm,

$$\|y\| = \sup_{x \in I} |y(x)|$$

Hence, X is a complete metric space.

Define $T: X \longrightarrow X$ by x.

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

For $y \in X$, $Ty \in X$, we have $(Ty)(x) = y(x)$.
It is clear that fixed point T are solns of the IVP.

Claim : The soln is unique.

Let $y_1, y_2 \in X$ and consider,

$$|(Ty_1)(x) - (Ty_2)(x)|$$
$$= \left| \int_{x_0}^{x} f(t, y_1(t)) - f(t, y_2(t))\, dt \right|$$
$$= k \int_{x_0}^{x} |y_1 - y_2|\, dt \qquad \{ f \text{ is Lipschitz} \}$$
$$= k\, |y_1 - y_2|\, (x - x_0)$$

$$\Rightarrow |(T^2 y_1)(x) - (T^2 y_2)(x)| \leq k \int_{x_0}^{x} |(Ty_1)(t) - (Ty_2)(t)|\, dt$$
$$\leq k^2 \int_{x_0}^{x} |y_1 - y_2|\, (t - x_0)\, dt$$

$$= K^2 |y_1 - y_2| \frac{(x-x_0)^2}{2}$$

There by,

$$|(T^n y_1)(x) - (T^n y_2)(x)| \le K^n |y_1 - y_2| \frac{(x-x_0)^n}{n!}$$

$$\le \frac{(Kh)^n}{n!} |y_1 - y_2|$$

$$\therefore \quad \| T^n y_1 - T^n y_2 \| \le \frac{(Kh)^n}{n!} |y_1 - y_2|$$

If we choose n large enough so that,

$$\beta = \frac{(Kh)^n}{n!} < 1$$

$$\therefore \quad \| T^n y_1 - T^n y_2 \| \le \beta \| y_1 - y_2 \|, \quad \beta < 1$$

$\Rightarrow$ T^n is a contradiction.

Hence by Banach fixed pt theorem, T has a unique fixed point. Hence, IVP has a unique soln.

Theorem :

Let,

(i) $y' = f(x,y)$; $y(x_0) = y_0$ &

(ii) $y' = f(x,y)$; $y(x_0^*) = y_0^*$

be two IVP for $a \le x \le b$ with the solution, $y(x) = y(x, x_0, y_0)$ &

$y^*(x_0) = y(x, x_0^*, y_0^*)$

Let $f \in Lip(D, K)$ & $|f(x,y)| \le M$, $\forall x \in [a,b]$ in D.

Then, given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$|y(x) - y^*(x)| < \varepsilon$ whenever,

$|x_0 - x_0^*| < \delta$ and

$|y_0 - y_0^*| < \delta$.

Proof :

Let $x_0, x_0^* \in [a,b]$, $x_0^* > x_0$ and we have $y(x_0) = y_0$ and $y(x_0^*) = y_0^*$.

Given that $f \in Lip(D, K)$ and $|f| \le M$.

Then by Picard's theorem, $\exists\ y(x)$ and $y^*(x)$ such that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y^*(x) = y_0^* + \int_{x_0}^{x} f(t, y^*(t))\, dt$$

$$y(x) - y^*(x) = y_0 - y_0^* + \int_{x_0}^{x} f(t, y(t))\, dt - \int_{x_0^*}^{x} f(t, y^*(t))\, dt$$

$$\Rightarrow y_0 - y_0^* + \int_{x_0}^{x_0^*} f(t, y(t))\, dt + \int_{x_0^*}^{x} f(t, y(t))\, dt - \int_{x_0^*}^{x} f(t, y^*(t))\, dt$$

Since $f \in Lip(D, K)$, we have,

$$|f(t, y(t)) - f(t, y^*(t))| \le K |y(t) - y^*(t)|$$

and $|f| \le M$.

Therefore,

$$|y(x) - y^*(x)| \le |y_0 - y_0^*| + M |x_0 - x_0^*| + K \int_{x_0^*}^{x} |y(t) - y^*(t)|\, dt$$

By Grönwall's inequality,

$$|y(x) - y^*(x)| \le \left[|y_0 - y_0^*| + M |x_0 - x_0^*| \right] e^{\int_{x_0^*}^{x} K\, dt}$$

$$\le [\delta + M\delta]\, e^{K(x - x_0^*)}$$

$$\le \delta(1+M)\, e^{K(b-a)}$$

Feb 16, Lect - 15

Generalised Gronwall's Inequality

If f, g and h are non-negative contin. functions defined for $x \in I$, then the inequality,

$$f(x) \le h(x) + \int_{x_0}^{x} g(t).\, f(t).\, dt, \quad x \ge x_0, \; x_0, x \in I.$$

$$\Rightarrow \quad f(x) \le h(x) + \int_{x_0}^{x} g(t).\, h(t)\, e^{\int_t^x g(s).ds}\, dt$$

Proof:

Given that,

$$f(x) \le h(x) + \int_{x_0}^{x} g(t).\, f(t).\, dt$$

$$= h(x) + z(x)$$

when $z(x) = \int g(t).\, f(t).\, dt$

$$\Rightarrow \quad f(x) \le h(x) + z(x) \quad \text{—— } (1)$$

$$f(x) - h(x) \le z(x) \quad \text{—— } (2)$$

$$z'(x) = g(x)\, f(x) \quad \text{—— } (3)$$

Since g is non-negative, from (1),

$$g(x).\, f(x) \le g(x).\, h(x) + g(x).\, z(x)$$

$$z'(x) \le g(x).\, h(x) + g(x).\, z(x)$$

$$z'(x) - g(x).\, z(x) \le g(x).\, h(x).$$

Using the IF, $e^{-\int_{x_0}^{x} g(t).dt}$, we have,

$$z(x).\, e^{-\int_{x_0}^{x} g(t).dt} \le \int_{x_0}^{x} g(t).\, h(t).\, e^{-\int_{x_0}^{t} g(s).ds}\, dt$$

$$\Rightarrow \quad z(x) \le \int_{x_0}^{x} g(t).\, h(t).\, e^{\int_{x_0}^{x} g(t).dt - \int_{x_0}^{t} g(s).ds}.\, dt$$

$$= \int_{x_0}^{x} g(t).\, h(t).\, e^{\int_t^x g(s).ds}$$

From (2), we have,

$$f(x) - h(x) \le z(x) \le \int_{x_0}^{x} g(t).\, h(t).\, e^{\int_t^x g(s).ds}.\, dt$$

$$\Rightarrow \quad f(x) \le h(x) + \int g(t).\, h(t).\, e^{\int_t^x g(s).\, h(s)\, dx}$$

Feb 18, Lect -17 (?)

Continuity of the soln. depends on the Dynamics

Theorem:

Let $R = \{ (x, y) : |x - x_0| \le a,\; |y - y_0| \le b,\; a, b > 0 \}$

Let f, g be two continuous funcs. on R satisfying,

(i) $f \in Lip[R, K]$ with respect to y.

(ii) given $\varepsilon > 0$, $|f(x,y) - g(x,y)| < \varepsilon, \; \forall\, x, y \in R$

Let $y_1(x)$ and $y_2(x)$ be the soln of the following IVP with (x, y_1) & $(x, y_2) \in R$ for $|x - x_0| < h$.

$$y'(x) = f(x,y) \quad ; \quad y(x_0) = y_0$$

$$y'(x) = g(x,y) \quad ; \quad y(x_0) = y_0$$

Then $|y_1(x) - y_2(x)| \le \frac{\varepsilon}{K}\left(e^{Kh} - 1 \right)$, $h = x - x_0$.

Proof:

Given that,

$$y' = f(x,y), \quad y(x_0) = y_0 \quad \text{—— } (1)$$

$$y' = g(x,y), \quad y(x_0) = y_0 \quad \text{—— } (2)$$

and y_1, and y_2 are solns of (1) & (2) resp

Let,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \quad \text{and}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} g(t, y_2(t))\, dt$$

$$y_1(x) - y_2(x) = \int_{x_0}^{x} f(t, y_1(t))\,dt - \int_{x_0}^{x} g(t, y_2(t))\,dt$$

$$= \int_{x_0}^{x} \left(f(t, y_1(t)) - \underline{f(t, y_2(t))}\right) dt$$

$$+ \int_{x_0}^{x} \left(\underline{f(t, y_2(t))} - g(t, y_2(t))\right) dt$$

$$|y_1 - y_2| \le \int_{x_0}^{x} \left|f(t, y_1(t)) - f(t, y_2(t))\right| dt \; +$$

$$\int_{x_0}^{x} \left| f(t, y_2(t)) - g(t, y_2(t))\right| dt$$

$$\le K \int_{x_0}^{x} |y_1(t) - y_2(t)|\, dt + \varepsilon .(x - x_0)$$

$$\longrightarrow (3)$$

Let $f = |y_1 - y_2|$, $g = K$ and $h = \varepsilon(x - x_0)$ in generalised Grönwall inequality.

Hence by the same theorem,

$$|y_1(x) - y_2(x)| \le \varepsilon(x - x_0) + \int_{x_0}^{x} \varepsilon K(t - x_0).\, e^{\int_t^x K.ds}\, dt$$

$$= \varepsilon(x - x_0) + \varepsilon K \int_{x_0}^{x} (t - x_0) e^{K(x-t)}\, dt$$

$$= \varepsilon(x - x_0) + \varepsilon K \int_{x_0}^{x} (t - x_0) e^{-K(t-x)}\, dt$$

$$= \varepsilon(x - x_0) - \varepsilon \left[(t - x_0).\, e^{-K(t-x)} + \frac{e^{-K(t-x)}}{K} \right]_{x_0}^{x}$$

$$= -\frac{\varepsilon}{K}\left(1 - e^{K(x - x_0)}\right)$$

$$= \frac{\varepsilon}{K}\left(e^{Kh} - 1\right)$$

Continuity of soln. depends on the initial conditions and dynamics of the soln.

Theorem:

Let $R = \{(x,y) : |x - x_0| \le a,\ |y - y_0| \le b,\ a, b > 0\}$

Suppose $f, g \in C(R)$ and be lipschitz continuous with respect to y on R with lips. constants k_1 and k_2 respectively.

Let $y_1(x)$ and $y_2(x)$ be respectively the soln of IVP.

$$y'(x) = f(x,y), \qquad y(x_0) = y_0$$

$$y'(x) = g(x,y), \qquad y(x_0^*) = y_0^*$$

in some interval I_1, I_2 containing x_0, x_0^* where both y_1 and y_2 are defined.

Then,

$$\max_{x \in I} |y_1(x) - y_2(x)| \le \Big[|y_0 - y_0^*| + |I| \max_{R} |f(x,y) - g(x,y)| + M |x_0 - x_0^*| \Big] e^{K_0 |I|}$$

where l is the length of this interval,

$$M = \max\left\{ \max_{x \in R} f,\ \max_{x \in R} g \right\}$$

$$K_0 = \min\{K_1, K_2\}$$

Proof:

We have,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt$$

$$y_2(x) = y_0^* + \int_{x_0^*}^{x} g(t, y_2(t))\, dt$$

$$(y_1 - y_2)(x) = (y_0 - y_0^*) + \int_{x_0}^{x} f(t, y_1(t))\, dt - \int_{x_0^*}^{x} g(t, y_2(t))\, dt$$

$$= (y_0 - y_0^*) + \int_{x_0}^{x} (f(t, y_1(t)) - f(t, y_2(t)))\, dt$$

$$+ \int_{x_0}^{x} f(t, y_2(t))\, dt - \int_{x_0^*}^{x} g(t, y_2(t))\, dt$$

$$|y_1 - y_2| \le |y_0 - y_0^*| + K_1 \int_{x_0}^{x} |y_1 - y_2|\, dt + \int_{x_0}^{x_0^*} |f(t, y_2^*)|\, dt$$

$$+ \int_{x_0^*}^{x} |f(t, y_2(t)) - g(t, y_2(t))|\, dt$$

$$\le |y_0 - y_0^*| + M|x_0^* - x_0|$$

$$+ |x - x_0^*| |f(t, y_2(t)) - g(t, y_2(t))|$$

$$+ K_1 \int_{x_0}^{x} |y_1(t) - y_2(t)|\, dt$$

$$\max_{x \in R} |y_1 - y_2| \le |y_0 - y_0^*| + M|x_0^* - x_0|$$

$$+ |I| . \max |f(x,y) - g(x,y)|$$

$$+ K_1 \int_{x_0}^{x} |y_1(t) - y_2(t)|\, dt$$

By Grönwall's inequality,

$$\max |y_1 - y_2| \le [|y_0 - y_0^*| + M|x_0^* - x_0| +$$

$$|I| . \max \{|f - g|\}\}$$

repeat the process by multiplying with g,

we get $K_2 m (x_0^* - x_0)$

Thus,

$$\max_{x \in I} |y_1(x) - y_2(x)| \le \left[|y_0 - y_0^*| + M_1 \right.$$

$$\left. + |I| . \max_R |f(x,y) - g(x,y)| \right]$$

Feb 19, Lect - 18

$$y' = x + y$$

$$y(0) = 0$$

$$R : \{(x,y) : |x| \le 1, |y| \le 1\}$$

$$M = 2, \quad K = 1, \quad h = \min\{a, \tfrac{b}{M}\} = \tfrac{1}{2}$$

$|x - x_0| \le h$: nhbd of existence, local existence — soln exists locally.

$$[x_0 - h, x_0 + h]$$

$[-\frac{1}{2}, \frac{1}{2}] \to$ interval of existence.

If the soln exists in the interval $|x - x_0| \le a$ then we say that soln exist non-locally.

If I_1 and I_2 are interval of existence, the $I_1 \cup I_2$ is also an interval of existence. $\{I_i\}$ is a collective of interval of existence, then $J = \cup I$ is called maximal interval of existence.

$$y' = y^2$$

$$y(1) = -1$$

$$R = \{(x,y) : |x-1| \le 1, |y-1| \le 1\}$$

$$h = 1/4.$$

int. of exist $= \left[\frac{3}{4}, \frac{5}{4}\right]$.

Theorem:

Suppose $f(x,y)$ be a bold continuous func. defined on a domain and f is Lipschitz in y on the domain.

Then, the largest open interval over which the solution $y(x)$ with $y(x_0)$ of the IVP.

$$y' = f(x,y) \quad ; \quad y(x_0) = y_0$$

defined in any one of the following two types:

(i) (a, b) where both a, b are finite, or either a or b is finite.

(ii) the entire x axis. $-\infty < x < \infty$.

Proof :

Let $R = \{(x,y) : |x-x_0| \le a \; ; \; |y-y_0| \le b\}$

Given that, $y' = f(x,y)$

$y(x_0) = y_0.$

We have, $(x_0, y_0) \in R$ and by Picards theorem, $\exists$ a soln. Let it be ϕ_0.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt = \phi_0.$$

$$x \in [x_0-h, x_0+h]$$

Let $x_1 = x_0 + h$ and $y_1 = \phi_0(x_1)$

Clearly $(x_1, y_1) \in R$.

Reapply Picard's theorem to

$y' = f(x,y)$

$y(x_1) = y_1.$

There exist a solution say ϕ_1 such that

$\phi_1(x_1) = y_1$ in $[x_1, x_1+h_1]$, $h_1 > 0$.

Define
$$y(x) = \begin{cases} \phi_0(x) \; ; & x_0 - h \le x \le x_0 + h \\ \phi_1(x) \; ; & x_1 \le x \le x_1 + h_1 \end{cases}$$

Claim y is a solution.

Clearly, y is continuous.

In $[x_0-h, x_0+h]$, we have,

$$y(x) = \phi_0(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt$$

In $[x_1, x_1+h_1]$, we have

$$y(x) = \phi_1(x) = y_1 + \int_{x_1}^{x} f(t, \phi_1(t))\, dt$$

i.e., $y(x) = y_1 + \int_{x_1}^{x} f(t, y(t))\,dt$

But, $y_1 = \phi_0(x_1) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$

$$\therefore \; y(x) = y_0 + \int_{x_0}^{x_1} f(t, y(t))\, dt + \int_{x_1}^{x} f(t, y(t))\, dt$$

$$= y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \; , \quad [x_0-h, x_0+h]$$

Continue the process, we can extend the solution to $[x_0-h, x_n + h_n]$, $h_n \ge 0$

Apply the above process to the left side of the interval $[x_0-h, x_0+h]$, we get an interval $[a_n, b_n]$ such that,

$$[x_0-h, x_0+h] \subseteq [a_1, b_1] \subseteq [a_2, b_2] \dots \subseteq [a_n, b_n]$$

Let $a = \lim_n a_n$

$b = \lim_n b_n$

whether limit exists finitely or infinitely. In either case, we can extend the interval to an (a,b) where (a,b) is of the following form,

$$(-\infty, b), (a, \infty), (a,b) \text{ or } (-\infty, \infty)$$

depends on the limit.

Theorem :

Let f be a real cont. function on the strip,

$$S = \{(x,y) \; ; \; |x-x_0| \le a, \; |y| < \infty, \; a > 0\}$$

and f satisfy the Lipschitz condition with constant $k > 0$. Then the successive approximation,

$\{y_n\}$ of $y' = f(x,y)$; $y(x_0) = y_0$

exists on the existence interval $|x-x_0| \le a$ and converges to a soln $y(x)$ of the IVP.

Proof :

We note that the region is a strip and f need not be bounded on S. Since f is not bdd. the procedure that we followed in Picards theorem to prove the convergence is not applicable.

But ϕ_n is the n^{th} partial sum of the series,

$$\phi_0 + \sum_1^{\infty} (\phi_i - \phi_{i-1})$$

Since f is not bdd, we follow a diff. approach to prove that the series convg.

Let $M_0 = |y_0|$ and $M_1 = \max |\phi_1(x)|$

Claim that M_0 and M_1 are well defined. Since y_0 is given to us, M_0 is well defined. For a fixed y_0, $f(x, y_0)$ is cont on $|x - x_0| \leq a$

Hence, $\phi_1(x) = \phi_0 + \int_{x_0}^{x} f(t, y_0)\, dt$ is cont.

on $|x - x_0| \leq a$.

$\Rightarrow$ ϕ_1 attains maxim. on $|x - x_0| \leq a$

$\Rightarrow$ $M_1 = \max |\phi_1(x)|$ is well defined.

let $M = M_0 + M_1$.

Then,

$$|\phi_0| \leq M \quad \text{and} \quad |\phi_1 - \phi_0| \leq M.$$

$$|\phi_2(x) - \phi_1(x)| = \left| \int_{x_0}^{x} f(t, \phi_1) - f(t, \phi_0)\, dt \right|$$

$$\leq K \int_{x_0}^{x} |\phi_1 - \phi_0|\, dt = K.M(x - x_0)$$

$$|\phi_3(x) - \phi_2(x)| = \left| \int_{x_0}^{x} (f(t, \phi_2) - f(t, \phi_1))\, dt \right|$$

$$|\phi_3(x) - \phi_2(x)| \leq \frac{K^2 M (x - x_0)^2}{2}$$

In general,

$$|\phi_{n+1}(x) - \phi_n(x)| \leq \frac{M K^n (x - x_0)^n}{n!}$$

By applying Weierstrass method similar to Picard's theorem, we see that same convg. to $\phi(x)$.

Claim ϕ is a solution.

$$\phi(x) - y_0 - \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$$= \phi(x) - \left[\phi_n(x) - \int_{x_0}^{x} f(t, \phi_n(t))\, dt \right] - \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$$\leq |\phi - \phi_n| + \int_{x_0}^{x} |f(t, \phi_n) - f(t, \phi)|\, dt$$

$$\leq |\phi - \phi_n| + K \int_{x_0}^{x} |\phi_n - \phi|\, dt$$

Since $\phi_n \to \phi$ uniformly, the RHS $\longrightarrow 0$ as $n \longrightarrow \infty$.

$$\Rightarrow \quad \phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

Hence ϕ is a solution of the IVP.