Lecturer: Shrihari Sridharan
MAT404/MSM504 Topology
Mid-semester Examination

Max Score: 30 | February 27, 2026 | Duration: 120 minutes

General Instructions

- This question paper contains five questions.
- The individual marks for every question is as written before the start of the question or the appropriate part of the question.
- Please ensure to write the appropriate question number and its part number clearly, in the margin, before you begin your answer to any question or its part.
- Kindly explain your answers adequately well. Cryptic answers without any clarity shall be struck down without any marks.

Questions

1. Consider the collection of all semi-open intervals L of $\mathbb{R}$ given by

$$L \quad = \quad \Big\{[a,\ b)\ :\ a, b \in \mathbb{R}\ \text{ and }\ a < b\Big\}.$$

(a) (2 marks) Prove: L is a basis for a topology on $\mathbb{R}$.

Suppose $\mathscr{T}_L$ denotes the topology generated by L and $\mathscr{T}$ denotes the standard topology on $\mathbb{R}$.

(b) (1 mark) Compare the topologies $\mathscr{T}_L$ and $\mathscr{T}$.

Let $L_{\mathbb{Q}}$ be given by

$$L_{\mathbb{Q}} \quad = \quad \Big\{[p,\ q)\ :\ p, q \in \mathbb{Q}\ \text{ and }\ p < q\Big\}.$$

(c) (1 mark) Is $L_{\mathbb{Q}}$ a basis for $\mathscr{T}_L$? Justify.

(d) (2 marks) Is $(\mathbb{R},\ \mathscr{T}_L)$ a connected space? Justify.

2. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f : X \longrightarrow Y$. Suppose $A \subseteq X$.

(a) (1 mark) When is f said to be continuous?

(b) (1 mark) If $f : X \longrightarrow Y$ is continuous, is the restriction of f on A, denoted by $f|_A : A \longrightarrow Y$ continuous? Justify.

Suppose $X = \mathbb{R}$ with the standard topology. Let $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by

$$g(x) \quad = \quad \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \quad \text{and} \quad h(x) \quad = \quad \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0. \end{cases}$$

(c) (1 mark) Is $g|_{(-\infty,0]}$ continuous? Justify.

(d) (1 mark) Is $g|_{(0,\infty)}$ continuous? Justify.

(e) (1 mark) Is g continuous on $\mathbb{R}$? Justify.

(f) (1 mark) Is $h|_{(-\infty,0]}$ continuous? Justify.

(g) (1 mark) Is $h|_{(0,\infty)}$ continuous? Justify.

(h) (1 mark) Is h continuous on $\mathbb{R}$? Justify.

3. (4 marks) Consider $\mathbb{R}$ equipped with the Euclidean metric and the discrete metric, denoted by d and Δ respectively and given by

$$d(x, y) = |x - y| \quad \text{and} \quad \Delta(x, y) \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Let $I : (\mathbb{R}, \Delta) \longrightarrow (\mathbb{R}, d)$ be the identity function given by $I(x) = x$. Verify if I is a homeomorphism.

4. Define the spaces $Y, Y^* \subsetneq \mathbb{R}^2$ as

$$Y = \Big([0, 1] \times \{0\}\Big) \bigcup \Big(C \times [0, 1]\Big), \quad \text{where } C = \left\{\frac{1}{n} : n \in \mathbb{Z}_+\right\}$$

$$Y^* = \bigg(Y \cup \Big(\{0\} \times [0, 1]\Big)\bigg) \setminus \Big\{(0, 0)\Big\}.$$

Prove the following statements.

(a) (2 marks) Y is path connected.

(b) (2 marks) Y^* is connected.

(c) (2 marks) Y^* is not path connected.

(d) (2 marks) $Y^* \cap \Big(\mathbb{R} \times \{1\}\Big)$ is totally disconnected.

5. Let (Z, d) be a metric space and $B \subseteq Z$. For any $z \in Z$, define a function $d_B : Z \longrightarrow \mathbb{R}$ given by $d_B(z) = \inf\Big\{d(z, b) : b \in B\Big\}$.

(a) (2 marks) Prove: $\Big|d_B(z_1) - d_B(z_2)\Big| \leq d(z_1, z_2), \forall z_1, z_2 \in Z$.

Fix $\delta > 0$ and define $K_\delta = \Big\{z \in Z : d_B(z) \geq \delta\Big\}$.

(b) (1 mark) Is the set K_δ open or closed in Z? Justify.

(c) (1 mark) Find $\lim\limits_{\delta \to 0} K_\delta$.